Classification

of

two-parameter bifurcations

by

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SUMMARY

This thesis contains the classification of two-parameter bifurcations up to codimension three, using a two-parameter version of parametrised contact equivalence.

Part one contains the classification up to codimension one. The result consists of the following components:

1. A list of normal forms for the germs having codimension less or equal to one.

2. Recognition conditions for each normal form in the list, i. e. conditions that characterise the equivalence class of the normal form. These conditions are equations and inequalities for the Taylor coefficients of the germs.

3. Universal unfoldings for each normal form.

The result is obtained by investigating the structure of the orbits, which are induced by the action of the group of equivalences on the space of all bifurcation problems. Techniques from algebra, algebraic geometry and singularity theory are applied.

In part two the classification is extended to codimension three. The second chapter of part two contains a generalisation of the singularity approach to equivariant bifurcation theory. The case of an action of a compact Lie group on state and parameter space is considered. The main example is the case of bifurcations with a certain $\mathbf{D}_4$-symmetry.

PREFACE

This thesis is divided into two parts. Each part contains its own introduction and list of references.

I thank my supervisor Dr. Ian Stewart for his support — mathematical and otherwise — while this thesis gradually came into existence. Furthermore, I thank Dr. Mark Roberts, Dr. Ian Melbourne and Dr. Ton Marar for some very helpful discussions. I am also grateful to Prof. Jim Damon for pointing out the reasoning in example II. 3. 6. 2 of part one. Finally, I would like to thank the University of Warwick for one year of financial support.

Part One

CONTENTS

Chapter V

Immer wenn uns
Die Antwort auf eine Frage gefunden schien
Löste einer von uns an der Wand die Schnur der alten
Aufgerollten chinesischen Leinwand, so daß sie herabfiel und
Sichtbar wurde der Mann auf der Bank, der
So sehr zweifelte.

Bertolt Brecht

CHAPTER 1

Introduction

Golubitsky and Schaeffer [6] used methods from singularity theory to study bifurcations. This involves defining an appropriate equivalence relation on the set of all bifurcation problems and classifying these up to some codimension. In [7], for example, the same authors classify one-parameter bifurcations up to codimension four using the notion of parametrised contact equivalence. This result was extended, for problems in one state variable, up to codimension seven by Keyfitz [9]. There is a multitude of other classifications — many for equivariant bifurcations.

The problem treated in this thesis is the classification of two-parameter bifurcations in one state variable up to codimension one, using a two-parameter version of parametrised contact equivalence. The result consists of the following components:

1. A list of normal forms for the germs having codimension less or equal to one.

2. Recognition conditions for each normal form in the list, i. e. conditions that characterise the equivalence class of the normal form. These conditions are equations and inequalities for the Taylor coefficients of the germs.

3. Universal unfoldings for each normal form.

In this context there is a result due to Izumiya [8], who considered germs of the form

$$x^2 + \varphi(\lambda_1, \lambda_2) \ . \tag{1.1}$$

where x is the state variable, λ_1 and λ_2 being the parameters. She classified these up to codimension five. As we shall show even at codimension zero there are germs which are not of the form (1.1), e. g.

$$x^3 + x\lambda_1 + \lambda_2 .$$

Izumiya does not give any recognition conditions.

The following is an outline of the contents of this thesis.

In chapter II we set up the theoretical basis for the methods used to obtain the classification: First we generalise the definition of parametrised contact equivalence to two-parameter bifurcations. The set of all such equivalences forms a group which acts on the space of all bifurcation problems. The equivalence classes are the orbits under this group action. For the classification it is necessary to characterise these orbits.

The first step is to show that for many germs this problem can be reduced to studying the action of an algebraic group on a finite dimensional vector space. In order to achieve this the concept of finite determinacy is used. A germ is called finitely determined if its equivalence class depends only on a finite number of its Taylor coefficients. Proving finite determinacy for a germ is more complicated than in the one-parameter case — the Malgrange-Mather Preparation Theorem has to be used.

The next step is to calculate the higher-order terms for certain normal forms, i. e. those terms which do not affect the equivalence class. Subsequently it is possible to determine the orbits under the group action modulo higher-order terms. To achieve this we use a theorem of Bruce, du Plessis and Wall [3], which guarantees that the orbits are algebraic varieties for *unipotent* equivalences. In order to apply this result we decompose the group of equivalences into a product, one of the factors being a subgroup of unipotent equivalences. Here we use the Bruhat decomposition for $GL(2, \mathbb{R})$. The decomposition can be generalised for equivalences of n-parameter bifurcations, since the Bruhat decomposition is valid for $GL(n, \mathbb{R})$.

To calculate the higher-order terms with respect to the unipotent equivalences, we use results developed for the one-parameter case by Melbourne [11]. According to one of his theorems, determining the higher-order terms is straightforward provided the equations defining the orbit are linear. Germs which satisfy this condition are called linearly determined. In the one-parameter case most germs of low codimension are linearly determined. This, however, is no longer true in the two-parameter case. Consequently, the calculations to determine the orbit become rather complicated. In this way the orbits of the normal forms are calculated with respect to the group of unipotent equivalences.

According to the decomposition of the group of equivalences the next step is to take scaling transformations into account. This is straightforward. Then the resulting recognition conditions have to be transformed into conditions with respect to the full group of equivalences. For several normal forms this turns out to be a non-trivial procedure. It involves finding certain polynomials which are invariant under the transformation. This is carried out in chapter III, section 4.

Knowing the list of recognition conditions immediately yields the classification. This is stated as theorem III. 6. 1.

Calculating the higher-order terms as described above involves selecting a particular normal form to start with. This leads to the question of how to reduce the amount of calculations by choosing the normal form in an appropriate way. We address this problem in chapter IV.

Chapter V contains a list of diagrams giving a geometrical description of the normal forms in the classification and their universal unfoldings.

CHAPTER II

1. Notation

We denote coordinates in $\mathbb{R} \times \mathbb{R}^2$ by x, λ_1, λ_2. Putting $\lambda := (\lambda_1, \lambda_2)$ we define $\mathcal{E}_{x,\lambda}$ to be the ring of all C^∞- function germs $\mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ at $(0, 0) \in \mathbb{R} \times \mathbb{R}^2$. $\mathcal{M}_{x,\lambda}$ denotes the maximal ideal in $\mathcal{E}_{x,\lambda}$.

Analogously defined are the rings $\mathcal{E}_x$ and $\mathcal{E}_\lambda$ and their maximal ideals $\mathcal{M}_x$ and $\mathcal{M}_\lambda$. Sometimes we abbreviate $\mathcal{M}_{x,\lambda}$ to $\mathcal{M}$.

Let V be a vector space over the field of real numbers and let $v_1, \ldots, v_k \in V$. Then $\mathbb{R}\{v_1, \ldots, v_k\}$ denotes the linear span of $v_1, \ldots, v_k$.

Let G be a Lie group. We denote its Lie algebra by *LG*.

$\mathbb{R}^{>0}$ denotes the multiplicative group of positive real numbers.

The function $sg: \mathbb{R} \longrightarrow \mathbb{R}$ is defined by

$$sg(x) := \begin{cases} +1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

Let h be a germ in $\mathcal{E}_{x,\lambda}$. We denote its Taylor coefficients as follows:

$$h_{x^\alpha \lambda_1^\beta \lambda_2^\gamma} := \frac{\partial^{\alpha+\beta+\gamma} h}{\partial x^\alpha \partial \lambda_1^\beta \partial \lambda_2^\gamma}(0, 0, 0) \quad .$$

For small values of α, β and γ we write h_x, h_{xx}, $h_{\lambda_j \lambda_j}$, $h_{x \lambda_j \lambda_j}$ etc., instead. It will always be clear from the context, whether $h = 0$ means $h(0) = 0$.

The following typographical scheme has been adopted throughout the text: All theorems and definitions are italicised and symbols and terms which are defined are printed in bold face, when they appear for the first time.

2. Parametrised contact equivalence

In this section we define parametrised contact equivalence for two-parameter bifurcations. This definition is analogous to the one introduced by Golubitsky and Schaeffer in the one-parameter case (See [6] and [7].). For later use two slightly modified versions of this equivalence relation are introduced. Each equivalence relation corresponds to a group and we state some results dealing with relations between these.

To avoid repetition the following definition incorporates all the three different equivalence relations. For notational convenience we use the term E-equivalence for parametrised contact equivalence. Compare [1], [4], [5] and [10] for the concept of ordinary contact equivalence.

2.1 Definition. *Two germs $f, g \in \mathcal{M}_{x,\lambda}$ are called E-equivalent , if there exist smooth germs $S, X\colon \mathbb{R}^3, 0 \to \mathbb{R}$, and $\Lambda_1, \Lambda_2\colon \mathbb{R}^2, 0 \to \mathbb{R}$ such that*

$$g(x, \lambda_1, \lambda_2) = S(x, \lambda_1, \lambda_2)\, f(X(x, \lambda_1, \lambda_2), \Lambda_1(\lambda_1, \lambda_2), \Lambda_2(\lambda_1, \lambda_2))$$

and the following conditions are satisfied:

$$\begin{aligned} X(0, 0, 0) &= 0 \\ \Lambda_1(0, 0) &= 0 \\ \Lambda_2(0, 0) &= 0 \\ S(0, 0, 0) &> 0 \\ X_x(0, 0, 0) &> 0\,; \end{aligned} \qquad (2.1)$$

$$\begin{vmatrix} (\Lambda_1)_{\lambda_1} & (\Lambda_1)_{\lambda_2} \\ (\Lambda_2)_{\lambda_1} & (\Lambda_2)_{\lambda_2} \end{vmatrix} \neq 0 \,. \tag{2.2}$$

Furthermore, if the germs X, Λ_1, Λ_2 and S satisfy the conditions (2.1) and additionally

$$\begin{aligned} S(0) &= 1 \\ X_x(0) &= 1 \\ (\Lambda_1)_{\lambda_1} &= 1 \\ (\Lambda_2)_{\lambda_1} &= 0 \\ (\Lambda_2)_{\lambda_2} &= 1 \end{aligned} \tag{2.3}$$

f and g are called ***U-equivalent*** *.*
Should X, Λ_1, Λ_2 and S satisfy (2.1), (2.3) and

$$(\Lambda_1)_{\lambda_2} = 0 \tag{2.4}$$

f and g are called ***$\hat{U}$-equivalent*** *.*

Let E be the set of all quadruples $(S, X, \Lambda_1, \Lambda_2)$ satisfying the conditions (2.1) and (2.2). E acts on $\mathcal{M}_{x,\lambda}$ in the following way: Let $f \in \mathcal{M}_{x,\lambda}$ and $e = (S, R) \in E$, where $R = (X, \Lambda_1, \Lambda_2)$. The conditions in the previous definition imply that R is a diffeomorphism germ $\mathbb{R}^3, 0 \longrightarrow \mathbb{R}^3, 0$. Then the action is defined by

$$e \,.\, f := S \cdot (f \circ R)\,. \tag{2.5}$$

E can be given a group structure by the following definition of a multiplication: Let $e_1 = (S_1, R_1)$, $e_2 = (S_2, R_2) \in E$. Then

$$e_2 \cdot e_1 := (S_2 \cdot S_1 \circ R_2, R_1 \circ R_2)$$

With this definition of multiplication formula (2.5) defines a group action of E on $\mathcal{M}_{x,\lambda}$. The orbits generated by this action are precisely the equivalence classes corresponding to E-equivalence.

Let U (respectively $\hat{U}$) be the set of all quadruples $(S, X, \Lambda_1, \Lambda_2)$ satisfying conditions (2.1) and (2.3) (respectively (2.1), (2.3) and (2.4)). Then the multiplication on E induces one on U and $\hat{U}$ each. In this way U and $\hat{U}$ become subgroups of E. Again, the orbits generated by the actions of U and $\hat{U}$ on $\mathcal{M}_{x,\lambda}$ correspond to the U- and $\hat{U}$-equivalence classes, respectively.

To illustrate the difference between these various equivalence relations, we consider the linear part of an element $e = (S, X, \Lambda_1, \Lambda_2) \in E$, where

$$\begin{aligned}
X(x, \lambda_1, \lambda_2) &= p\,x + q\,\lambda_1 + r\,\lambda_2 + \dots \\
\Lambda_1(\lambda_1, \lambda_2) &= \phantom{p\,x +{}} s\,\lambda_1 + t\,\lambda_2 + \dots \\
\Lambda_2(\lambda_1, \lambda_2) &= \phantom{p\,x +{}} u\,\lambda_1 + v\,\lambda_2 + \dots \;;
\end{aligned}$$

$$S(x, \lambda_1, \lambda_2) = A + B\,x + C\,\lambda_1 + D\,\lambda_2 + \dots$$

and $p, A > 0$.

If e is in U, the linear part reduces accordingly:

$$\begin{aligned}
X(x, \lambda_1, \lambda_2) &= x + q\,\lambda_1 + r\,\lambda_2 + \dots \\
\Lambda_1(\lambda_1, \lambda_2) &= \phantom{x +{}} \lambda_1 + t\,\lambda_2 + \dots \\
\Lambda_2(\lambda_1, \lambda_2) &= \phantom{x + \lambda_1 +{}} \lambda_2 + \dots \;;
\end{aligned}$$

$$S(x, \lambda_1, \lambda_2) = 1 + B\,x + C\,\lambda_1 + D\,\lambda_2 + \dots .$$

If e is in $\hat{U}$, we have

$$\begin{aligned} X(x,\lambda_1,\lambda_2) &= x + q\lambda_1 + r\lambda_2 + \dots \\ \Lambda_1(\lambda_1,\lambda_2) &= \qquad \lambda_1 \qquad + \dots \\ \Lambda_2(\lambda_1,\lambda_2) &= \qquad\qquad \lambda_2 + \dots \ ; \end{aligned}$$

$$S(x,\lambda_1,\lambda_2) = 1 + B\,x + C\lambda_1 + D\lambda_2 + \dots \ .$$

In the two latter cases the linear parts of S (i. e. $S(0)$) and $R = (X, \Lambda_1, \Lambda_2)$ are unipotent matrices. The groups of diffeomorphisms induced by U and $\bar{U}$ on $\mathcal{M}_{x,\lambda}$ are also unipotent. Therefore the theory for unipotent groups (see [7a]) of diffeomorphisms developed by Bruce, du Plessis and Wall (See [3].) applies to U and $\bar{U}$. One of their results will be stated in section 5.

In the remainder of this section we describe some properties of the groups of equivalences defined above. The first property is a decomposition of E. We introduce some notation: Let T, the group of scaling transformations, denote the subgroup of E consisting of all equivalences of the form

$$\begin{aligned} X(x,\lambda_1,\lambda_2) &= \nu x \\ \Lambda_1(\lambda_1,\lambda_2) &= \quad m\lambda_1 \\ \Lambda_2(\lambda_1,\lambda_2) &= \qquad\quad n\lambda_2 \ ; \end{aligned}$$

$$S(x,\lambda_1,\lambda_2) = \mu,$$

where $\mu, \nu > 0$ and $m, n \neq 0$. Let W denote the subgroup consisting of the identity and the equivalence given by

$$\begin{aligned} X(x,\lambda_1,\lambda_2) &= x \\ \Lambda_1(\lambda_1,\lambda_2) &= \qquad \lambda_2 \\ \Lambda_2(\lambda_1,\lambda_2) &= \quad \lambda_1 \ ; \end{aligned}$$

$$S(x,\lambda_1,\lambda_2) = 1 \ .$$

which interchanges λ_1 and λ_2 . Let N denote the subgroup consisting of all equivalences of the form

$$\begin{aligned} X(x,\lambda_1,\lambda_2) &= x \\ \Lambda_1(\lambda_1,\lambda_2) &= \lambda_1+\alpha\,\lambda_2 \\ \Lambda_2(\lambda_1,\lambda_2) &= \lambda_2 \ ; \end{aligned}$$

$$S(x,\lambda_1,\lambda_2) = 1 \ ,$$

where $\alpha \in \mathbb{R}$. Furthermore, let $B = T\,U$.

2.2 Proposition. *The group E can be decomposed as*

$$E = N\,W\,B = N\,W\,T\,U \ .$$

In order to show this, we need the Bruhat decomposition for $GL(2,\mathbb{R})$. Using the notation

$$B^{*} := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \;\middle|\; a\,d \neq 0 \right\},$$

$$N^{*} := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \;\middle|\; b \in \mathbb{R} \right\},$$

$$T^{*} := \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \;\middle|\; a\,d \neq 0 \right\},$$

$$W^{*} := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

this result is the following:

2.3 Proposition. *$GL(2, \mathbb{R})$ can be decomposed as*

$$GL(2, \mathbb{R}) = B^* W^* N^* \quad .$$

More precisely, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{R}) \setminus B^*$ *can be written as*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{bc-ad}{c} & a \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{d}{c} \\ 0 & 1 \end{pmatrix} . \tag{2.6}$$

A proof can be found in [2], for example. Note that the order of the factors in the decomposition is not the standard one but has been reversed.

Proof of proposition 2.2: Let $e = (S, X, \Lambda_1, \Lambda_2)$, where

$$\begin{aligned} X(x, \lambda_1, \lambda_2) &= p\,x + q\,\lambda_1 + r\,\lambda_2 + Q_1(x, \lambda_1, \lambda_2) \\ \Lambda_1(\lambda_1, \lambda_2) &= \qquad s\,\lambda_1 + t\,\lambda_2 + Q_2(\lambda_1, \lambda_2) \\ \Lambda_2(\lambda_1, \lambda_2) &= \qquad u\,\lambda_1 + v\,\lambda_2 + Q_3(\lambda_1, \lambda_2); \end{aligned}$$

$Q_1 \in M^2_{x,\lambda}$ and $Q_2, Q_3 \in M^2_{\lambda}$.

If $u = 0$ there is nothing to prove. Now suppose $u \neq 0$. Then $e = n\,w\,b$, where n is given by

$$X(x,\lambda_1,\lambda_2) = x$$
$$\Lambda_1(\lambda_1,\lambda_2) = \quad \lambda_1 + \frac{v}{u}\lambda_2$$
$$\Lambda_2(\lambda_1,\lambda_2) = \quad \lambda_2 \;;$$

$$S(x,\lambda_1,\lambda_2) = 1 \quad .$$

w by

$$X(x,\lambda_1,\lambda_2) = x$$
$$\Lambda_1(\lambda_1,\lambda_2) = \quad \lambda_2$$
$$\Lambda_2(\lambda_1,\lambda_2) = \quad \lambda_1 \;;$$

$$S(x,\lambda_1,\lambda_2) = 1 \quad .$$

and b = ($\underline{S}$, $\underline{X}$, $\underline{\Lambda}_1$, $\underline{\Lambda}_2$), where

$$\underline{X}(x,\lambda_1,\lambda_2) = p\,x + \left(r - \frac{v}{u}\,q\right)\lambda_1 + q\,\lambda_2 + Q_1\!\left(x, -\frac{v}{u}\lambda_1 + \lambda_2, \lambda_1\right)$$

$$\underline{\Lambda}_1(\lambda_1,\lambda_2) = \frac{t\,u - s\,v}{u}\lambda_1 + s\,\lambda_2 + Q_2\!\left(-\frac{v}{u}\lambda_1 + \lambda_2, \lambda_1\right)$$

$$\underline{\Lambda}_2(\lambda_1,\lambda_2) = u\,\lambda_2 + Q_3\!\left(-\frac{v}{u}\lambda_1 + \lambda_2, \lambda_1\right) \;;$$

$$\underline{S}(x,\lambda_1,\lambda_2) = S\left(x, -\frac{v}{u}\lambda_1 + \lambda_2, \lambda_1\right) \;.$$

This follows from proposition 2.3 . Obviously $n \in N$, $w \in W$ and $b \in B$.□

2.4 Remark. E is the disjoint union of B and the set

$$\left\{ (S, X, \Lambda_1, \Lambda_2) \in E \mid (\Lambda_2)_{\lambda_1} \neq 0 \right\} ,$$

and the elements of the latter set can be decomposed as described in the preceding proof.

The next statement is a decomposition of U.

2.5 Proposition. *The group U can be decomposed as*

$$U = N \hat{U} \quad .$$

Proof: Write $u = (S, X, \Lambda_1, \Lambda_2)$, where

$$\begin{aligned} X(x, \lambda_1, \lambda_2) &= x + q\,\lambda_1 + r\,\lambda_2 + Q_1(x, \lambda_1, \lambda_2) \\ \Lambda_1(\lambda_1, \lambda_2) &= \lambda_1 + t\,\lambda_2 + Q_2(\lambda_1, \lambda_2) \\ \Lambda_2(\lambda_1, \lambda_2) &= \lambda_2 + Q_3(\lambda_1, \lambda_2), \end{aligned}$$

$Q_1 \in \mathcal{M}^2_{x,\lambda}$ and $Q_2, Q_3 \in \mathcal{M}^2_{\lambda}$. Then $u = n\,\hat{u}$, where n is given by

$$\begin{aligned} X(x, \lambda_1, \lambda_2) &= x \\ \Lambda_1(\lambda_1, \lambda_2) &= \lambda_1 + t\,\lambda_2 \\ \Lambda_2(\lambda_1, \lambda_2) &= \lambda_2 \quad ; \end{aligned}$$

$$S(x, \lambda_1, \lambda_2) = 1 \quad .$$

and $\underline{u} = (\underline{S}, \underline{X}, \underline{\Lambda}_1, \underline{\Lambda}_2)$, where

$$\begin{aligned}
\underline{X}(x, \lambda_1, \lambda_2) &= x + q\lambda_1 + (r - qt)\lambda_2 + Q_1(x, \lambda_1 - t\lambda_2, \lambda_2) \\
\underline{\Lambda}_1(\lambda_1, \lambda_2) &= \lambda_1 + Q_2(\lambda_1 - t\lambda_2, \lambda_2) \\
\underline{\Lambda}_2(\lambda_1, \lambda_2) &= \lambda_2 + Q_3(\lambda_1 - t\lambda_2, \lambda_2);
\end{aligned}$$

$$\underline{S}(x, \lambda_1, \lambda_2) = S(x, \lambda_1 - t\lambda_2, \lambda_2) \quad . \square$$

2.6 Proposition. *$\tilde{U}$ is a normal subgroup of E.*

Proof: The reasoning is analogous to the one given by Melbourne in [11] for the one-parameter case. Mapping $(S, X, \Lambda_1, \Lambda_2)$ to

$$\left(S(0), X_x(0), \begin{bmatrix} (\Lambda_1)_{\lambda_1} & (\Lambda_1)_{\lambda_2} \\ (\Lambda_2)_{\lambda_1} & (\Lambda_2)_{\lambda_2} \end{bmatrix} \right)$$

defines a group homomorphism from E onto $\mathbb{R}^{>0} \times \mathbb{R}^{>0} \times GL(2, \mathbb{R})$. $\tilde{U}$ is the kernel of this homomorphism and hence a normal subgroup of E.

An alternative proof is to check $e\,\tilde{U}\,e^{-1} \subset \tilde{U}$ for all $e \in E$. $\square$

3. Tangent spaces

It is a well known feature of singularity theory that questions of equivalence can be treated on an infinitesimal level. The crucial construction involved is the tangent space to an orbit generated by the group of equivalences. The different group actions defined in section 2 give rise to different tangent spaces to the group orbits. First we give a geometrical definition of these tangent spaces. (See [4], for example.):

3.1 Definition. *Let G be a subgroup of E, LG its Lie algebra and exp: LG $\longrightarrow$ G the exponential map. Then*

$$T(f,\ G) \;=\; \left\{ \frac{d}{dt} \left(exp(t\,A)\,.\,f \right)_{t=0} \,/\, A \in LG \right\}$$

is called the G - tangent space of f.

To calculate the U- and U-tangent spaces of a given germ we use the following algebraic formulae:

3.2 Proposition.

$$T(f,\hat{U}) = \mathcal{E}_{x,\lambda} \left\{ x\,f, \lambda_1 f, \lambda_2 f, \lambda_1 f_x\,, \lambda_2 f_x\,, x^2 f_x \right\} +$$
$$\mathcal{E}_{\lambda} \left\{ \lambda_1^2 f_{\lambda_1}, \lambda_1 \lambda_2 f_{\lambda_1}, \lambda_2^2 f_{\lambda_1}, \lambda_1^2 f_{\lambda_2}, \lambda_1 \lambda_2 f_{\lambda_2}, \lambda_2^2 f_{\lambda_2} \right\}$$

$$T(f, U) = \mathcal{E}_{x,\lambda} \left\{ x f, \lambda_1 f, \lambda_2 f, \lambda_1 f_x, \lambda_2 f_x, x^2 f_x \right\} +$$
$$\mathcal{E}_{\lambda} \left\{ \lambda_2 f_{\lambda_1}, \lambda_1^2 f_{\lambda_1}, \lambda_1^2 f_{\lambda_2}, \lambda_1 \lambda_2 f_{\lambda_2}, \lambda_2^2 f_{\lambda_2} \right\} .$$

Proof: The results follow from definition 3.1 and the definitions of the groups $\tilde{U}$ and U. □

3.3 Example. Consider $f = A x^3 + B x \lambda_2^2 + C \lambda_1$, where $A, B, C \neq 0$. Then

$$T(f, \tilde{U}) = \mathcal{M}^4 + < \lambda_1, \lambda_2 >^3 + \mathbb{R} \left\{ x^2 \lambda_1, x^2 \lambda_2, x \lambda_1 \lambda_2, x \lambda_1^2, x \lambda_1, \lambda_1^2, \lambda_1 \lambda_2, \lambda_2^2 \right\}$$

and

$$T(f, U) = \mathcal{M}^4 + < \lambda_1, \lambda_2 >^3 +$$
$$\mathbb{R} \left\{ x^2 \lambda_1, x^2 \lambda_2, x \lambda_1 \lambda_2, x \lambda_1^2, x \lambda_1, \lambda_1^2, \lambda_1 \lambda_2, \lambda_2^2, \lambda_2 \right\} .$$

In order to define the concept of codimension, we need another kind of tangent space.

3.4 Definition. *Let f be a germ in $\mathcal{M}_{x,\lambda}$. Then*

$$T_e(f, E) = \mathcal{E}_{x\lambda} \{ f, f_x \} + \mathcal{E}_{\lambda} \{ f_{\lambda_1}, f_{\lambda_2} \}$$

is called the extended E-tangent space of f.

$T_e(f, E)$ is the infinitesimal construction associated to unfoldings of the germ f. We omit the details, since they are analogous to the one-parameter case (See [7].[1]).

Note that all these tangent spaces are $\mathcal{E}_\lambda$-modules, but — in general — not $\mathcal{E}_{x\,\lambda}$-modules.

3.5 Definition. *Let f be a germ in $\mathcal{M}_{x,\lambda}$. The **codimension** of f denoted by cod f is the codimension of $T_e(f, E)$ as a vector subspace of $\mathcal{E}_{x,\lambda}$.*

Note that the codimension of f is finite if and only if either T(f, U) or $T(f, \dot{U})$ has finite codimension. This follows from

$$T_e(f, E) = T(f, \dot{U}) + \mathbb{R}\left\{ f, f_x, x f_x, f_{\lambda_1}, \lambda_1 f_{\lambda_1}, \lambda_2 f_{\lambda_1}, f_{\lambda_2}, \lambda_1 f_{\lambda_2}, \lambda_2 f_{\lambda_2} \right\}$$

and

$$T(f, U) = T(f, \hat{U}) + \mathbb{R}\cdot\left\{ \lambda_2 f_{\lambda_1} \right\} .$$

The first step to determine the codimension of a germ is to show that this codimension is finite. We deal with this matter in the next section. For both germs appearing in the next example, we already assume that they have finite codimension. Consequently, we can do all calculations modulo $\mathcal{M}^k$ for some $k \in \mathbb{N}$ and this is to be assumed in this example.

[1]Note, however, that the terminology in [7] is slightly different.

3.6 Example. 1. Consider $f = x^3 + x\lambda_1^2 + \lambda_2$. After some calculations it turns out that

$$T_e(f, E) = \mathcal{M}^2 + \langle \lambda_1, \lambda_2 \rangle + \mathbb{R}\{1\}$$

and hence

$$\frac{\mathcal{E}_{x,\lambda}}{T_e(f, E)} = \mathbb{R}\{x\}\ .$$

Therefore $\operatorname{cod} f = 1$.

2. Consider $f = x^4 + x\lambda_1 + \lambda_2$. Then

$$T_e(f, E) = \mathcal{E}_{x,\lambda}\left\{x^4 + x\lambda_1 + \lambda_2, 4x^3 + \lambda_1\right\} + \mathcal{E}_\lambda\{x, 1\}$$

$$= \mathcal{E}_{x,\lambda}\left\{-3x^4 + \lambda_2, 4x^3 + \lambda_1\right\} + \mathcal{E}_\lambda\{x, 1\}\ .$$

Define a homomorphism $\varphi\colon \mathcal{E}_{x,\lambda} \longrightarrow \mathcal{E}_x$ by

$$\begin{aligned} \varphi(x) &= x \\ \varphi(\lambda_1) &= -4x^3 \\ \varphi(\lambda_2) &= 3x^4 \end{aligned}$$

Then the following formula holds:

$$\operatorname{cod} f = \dim_{\mathbb{R}} \frac{\mathcal{E}_x}{\varphi\left(T_e(f, E)\right)} .$$

This follows from the fact that

$$\ker \varphi = \mathcal{E}_{x,\lambda} \left\{ -3x^4 + \lambda_2, 4x^3 + \lambda_1 \right\},$$

which ensures that φ induces an isomorphism between $\mathcal{E}_{x,\lambda}/T_e(f,E)$ and

$$\frac{\mathcal{E}_x}{\varphi\left(T_e(f, E)\right)} .$$

The following equality holds:

$$\begin{aligned} \varphi\left(T_e(f, E)\right) &= \varphi\left(\mathcal{E}_\lambda \left\{ x, 1 \right\}\right) \\ &= \mathcal{E}_{-4x^3, 3x^4} \left\{ x, 1 \right\} . \end{aligned}$$

Hence

$$\frac{\mathcal{E}_x}{\varphi\left(T_e(f, E)\right)} = \mathbb{R} \left\{ x^2 \right\} ,$$

which implies $\operatorname{cod} f = 1$. I am grateful to Jim Damon for the reasoning in this example.

The next example shows that the germ $f = x^4 + x^2\lambda_1 + x\lambda_2$ has infinite codimension. We treat a more general case.

3.7 Example. Consider $f = x^4 + x^2\varphi(\lambda_1, \lambda_2) + x\,\psi(\lambda_1, \lambda_2)$, where $\psi \in \mathcal{M}_\lambda$. Then f has infinite codimension. We show this in the following way:

$$T_e(f, E) = \mathcal{E}_{x,\lambda}\left\{x^4 + x^2\varphi(\lambda) + x\,\psi(\lambda)\,,\, 4x^3 + 2x\,\varphi(\lambda) + \psi(\lambda)\right\}$$
$$+\ \mathcal{E}_\lambda\left\{x^2\frac{\partial\varphi(\lambda)}{\partial\lambda_1} + x\frac{\partial\psi(\lambda)}{\partial\lambda_1}\,,\, x^2\frac{\partial\varphi(\lambda)}{\partial\lambda_2} + x\frac{\partial\psi(\lambda)}{\partial\lambda_2}\right\}\ .$$

This expression can be estimated algebraically:

$$T_e(f, E) \subset \mathcal{E}_{x,\lambda}\left\{x^4, x^2\varphi(\lambda), x\,\psi(\lambda), x^3, x\,\varphi(\lambda), \psi(\lambda)\right\}$$
$$+\ \mathcal{E}_\lambda\left\{x^2\frac{\partial\varphi(\lambda)}{\partial\lambda_1}, x\frac{\partial\psi(\lambda)}{\partial\lambda_1}, x^2\frac{\partial\varphi(\lambda)}{\partial\lambda_2}, x\frac{\partial\psi(\lambda)}{\partial\lambda_2}\right\}$$
$$\subset \mathcal{E}_{x,\lambda}\left\{x, \psi(\lambda)\right\}\ .$$

Let I denote the ideal $\mathcal{E}_{x,\lambda}\{\,x, \psi(\lambda)\,\}$. Suppose now that f has finite codimension. Then

$$\mathcal{M}^k \subset T_e(f, E) \subset I$$

holds for some number $k \in \mathbb{N}$. Considering the varieties $V(\mathcal{M}^k)$ and $V(I)$ in $\mathbb{C}^3$ corresponding to the ideals $\mathcal{M}^k$ and I, we obtain

$$\mathcal{V}(I) \subset \mathcal{V}\left(\mathcal{M}^{k}\right) = \left\{ (0,0,0) \right\} .$$

This, however, is impossible, since the dimension of $\mathcal{V}(I)$ is 1. Hence f has infinite codimension.

For use in the next section we introduce a name for a subspace of $T_e(f, E)$ which is an $\mathcal{E}_{x,\lambda}$-module.

3.8 Definition. *Let f be a germ in $\mathcal{M}_{x,\lambda}$. Then*

$$RT_e(f, E) = \mathcal{E}_{x,\lambda} \{ f, f_x \}$$

is called the ***restricted extended E-tangent space*** *of f.*

The following result will be needed later:

3.9 Proposition. *Let f be a germ in $\mathcal{M}_{x,\lambda}$ and $g \in E$. Then*

$$T(g \cdot f, \hat{U}) = g \cdot T(f, \hat{U}) .$$

Proof: Using definition 3.1 we obtain

$$T(g \cdot f, \hat{U}) = \left\{ \frac{d}{dt} \left(\exp(t A) \cdot (g \cdot f) \right)_{t=0} \,\middle|\, A \in L\hat{U} \right\}$$

$$= \left\{ \frac{d}{dt} \left(g \cdot \left(g^{-1} \exp(t\,A)\, g \right) \cdot f \right)_{t=0} \;\middle|\; A \in L\hat{U} \right\}$$

$$= g \cdot \left\{ \frac{d}{dt} \left(\left(g^{1} \exp(t\,A)\, g \right) \cdot f \right)_{t=0} \;\middle|\; A \in L\hat{U} \right\} . \qquad (3.1)$$

Since $\bar{U}$ is normal in E (See proposition II. 2. 6.) , $g^{1}\, \hat{U}\, g = \hat{U}$ holds for all $g \in E$. Therefore the curves $g^{-1} \exp(t\,A)\, g$ in $\bar{U}$ range over all curves in $\hat{U}$ through the identity element in $\bar{U}$. This implies that the expression in (3.1) is equal to

$$g \cdot \left\{ \frac{d}{dt} \left(\exp(t\,A) \cdot f \right)_{t=0} \;\middle|\; A \in L\hat{U} \right\}$$

$$= g \,.\, T(f, \hat{U}) \quad ,$$

which proves the result. □

4. Finite determinacy

Due to the mixed module structure of $T_e(f, E)$, proving finite-determinacy for two-parameter bifurcations is rather complicated. In the one-parameter case this problem can be circumvented: There it is sufficient to consider $RT_e(f, E)$, since this space has finite codimension if and only if $T_e(f, E)$ has. (This result, which is due to Damon, is stated in [7]). This is no longer true in the two-parameter case.

It is a theorem of Damon (theorem 10.2 in [3a]) that a germ f is finitely determined, if and only if it has finite codimension.

In the remainder of this section we abbreviate $T_e(f, E)$ and $RT_e(f, E)$ to $T_e(f)$ and $RT_e(f)$ respectively.

The following method will be used: For some appropriate pairs (k, ℓ) of non-negative integers the property

$$\mathcal{M}_\lambda^k \left(\mathcal{M}_x^\ell \, \mathcal{E}_{x,\lambda} \right) \subset T_e(f) \quad . \tag{4.1}$$

is verified. Here $T_e(f)$ and $\mathcal{M}_x{}^\ell \mathcal{E}_{x,\lambda}$ are regarded as $\mathcal{E}_\lambda$-modules. Instead of checking (4.1), we shall use the statement in proposition 4.2 below. First, though, it is necessary to introduce some more standard terminology from singularity theory, see [1], [4], [5] and [10].

4.1 Definition. *A germ $f_0 \in \mathcal{E}_x$ has finite K-codimension, if*

$$T_e K(f_0) := \mathcal{E}_x \left\{ f_0 , \ (f_0) \right\}$$

is of finite codimension as a subspace of $\mathcal{E}_x$.

4.2 Proposition. *Let $f \in \mathcal{E}_{x,\lambda}$ and $f_0 := f(x,0,0)$. If f_0 is of finite K-codimension then the condition*

$$\mathcal{M}_\lambda^k \left(\mathcal{M}_x^\ell \frac{\mathcal{E}_{x,\lambda}}{RT_e(f)} \right) \subset \frac{T_e(f)}{RT_e(f)} + \mathcal{M}_\lambda^{k+1} \left(\mathcal{M}_x^\ell \frac{\mathcal{E}_{x,\lambda}}{RT_e(f)} \right)$$

implies that

$$\mathcal{M}_\lambda^k \left(\mathcal{M}_x^\ell \mathcal{E}_{x,\lambda} \right) \subset T_e(f) \quad .$$

Proof: The statement is an immediate consequence of Nakayama's lemma, once it has been shown that $\mathcal{E}_{x,\lambda}/RT_e(f)$ is a finitely-generated $\mathcal{E}_\lambda$-module. We show this in the following way: Since f_0 has finite K-codimension, there exist germs $m_1, \ldots, m_k \in \mathcal{E}_x$ such that

$$\frac{\mathcal{E}_x}{T_eK(f_0)} = \mathbb{R}\ \{m_1, \ldots, m_k\} \quad .$$

Using the following isomorphism

$$\frac{\mathcal{E}_x}{T_eK(f_0)} \cong \frac{\mathcal{E}_{x,\lambda}}{RT_e(f) + \langle \lambda_1, \lambda_2 \rangle \mathcal{E}_{x,\lambda}}$$

we obtain

$$\frac{\mathcal{E}_{x,\lambda}}{RT_e(f) + \langle \lambda_1, \lambda_2 \rangle \mathcal{E}_{x,\lambda}} \cong \mathbb{R}\ \{m_1, \ldots, m_k\} \quad ,$$

or equivalently

$$\frac{\mathcal{E}_{x,\lambda}}{RT_e(f)} = \mathbb{R}\{m_1, \ldots, m_k\} + <\lambda_1, \lambda_2> \mathcal{E}_{x,\lambda}. \qquad (4.2)$$

Since $\mathcal{E}_{x,\lambda}/RT_e(f)$ is a finitely-generated $\mathcal{E}_{x,\lambda}$-module the following version of the Malgrange-Mather Preparation Theorem can be applied (See [10],p. 134):

4.3 Theorem. *Let M be a finitely-generated $\mathcal{E}_{x,\lambda}$-module, $m_1, \ldots, m_k \in \mathcal{E}_{x,\lambda}$, N an $\mathcal{E}_{x,\lambda}$-submodule of M and $\pi(x, \lambda) := \lambda$. Then the following conditions are equivalent:*

A) $N + \mathcal{E}_\lambda\{m_1, \ldots, m_k\} = M$

B) $N + \mathbb{R}\{m_1, \ldots, m_k\} + (\pi^* \mathcal{M}_\lambda) M = M$.

Here $\pi^* \mathcal{M}_\lambda$ denotes the ideal generated by the components of π.

Putting $M = \mathcal{E}_{x,\lambda}$ and $N = RT_e(f)$ it follows that condition (4.2) is equivalent to

$$\frac{\mathcal{E}_{x,\lambda}}{RT_e(f)} = \mathcal{E}_\lambda\{m_1, \ldots, m_k\} \quad ,$$

i. e. $\mathcal{E}_{x,\lambda}/RT_e(f)$ is a finitely-generated $\mathcal{E}_\lambda$-module. □

4.4 Example. Consider $f = x^4 + x\lambda_1 + \lambda_2$. Then f is 4-determined. This can be shown by the method described here or similarly as in the proof of lemma 3.5.3 in part two of this thesis.

5. Orbits of unipotent Subgroups of Equivalences

The following theorem of Bruce, du Plessis and Wall shows why it is useful to consider the unipotent subgroups of equivalences defined in section 2.

5.1 Theorem. *Let U be an unipotent affine algebraic group over $\mathbb{R}$ acting algebraically on a real affine algebraic variety V. Then the orbits of U are closed in the Zariski topology of V, i. e. they are real algebraic subvarieties of V.*

Proof: See [3].

5.2 Remark. If G is an algebraic group acting algebraically on a smooth algebraic variety, then the orbits are smooth semi-algebraic sets. See [4] for a proof of this fact. Under the assumptions of the preceding theorem and if V is smooth — in particular, if V is a finite-dimensional vector space — the orbits are smooth real algebraic subvarieties of V.

For one-parameter bifurcations the orbits of the groups of unipotent equivalences are in fact affine linear subspaces in many cases. This is shown by Melbourne in [11]. It will turn out that the situation is entirely different for two-parameter bifurcations.

CHAPTER III

1. Higher - order terms

We give the definition of higher-order terms and some results due to Melbourne (see [11]), who proved them for one-parameter bifurcations.

1.1 Definition. *Let f be a germ in* $\mathcal{M}_{x\,\lambda}$. *Then*

$$M(f, U) := \{ p \in \mathcal{M}_{x,\lambda} \mid f + p \in U.f \}$$

$$= \{ u.f - f \mid u \in U \} \quad .$$

Note that M(f, U) consists exactly of those germs that do not change the equivalence class of f when added to it. Hence determining M(f, U) solves the recognition problem. However, to do this in practice, we need a slightly different concept of higher-order terms. In order to define this we introduce some terminology first.

1.2 Definition. *Let G be a subgroup of E. A subspace V of* $\mathcal{M}_{x,\lambda}$ *is called* ***G–intrinsic****, if it is invariant under the action of G, i. e.* $G.V \subset V$ *. If a subset M of* $\mathcal{M}_{x,\lambda}$ *contains a unique maximal G-intrinsic subspace, then this is called the* ***G–intrinsic part*** *of M and is denoted by* $\mathbf{Itr}_G\, M$.

1.3 Definition. *Let U be a unipotent subgroup of E. We call*

$$P(f, U) := \{ p \in \mathcal{M}_{x,\lambda} \mid g + p \in U.f \text{ for all } g \in U.f \}$$

the module of U-higher-order terms.

1.4 Theorem. *Let f be a germ in* $\mathcal{M}_{x,\lambda}$ *of finite codimension and U a unipotent subgroup of E. Then*

A) $P(f, U) = Itr_U\, M(f, U)$,

B) $P(f, U) = Itr_U\, T(f, U)$.

Proof: See [11]. The facts that U is unipotent and its action on $\mathcal{M}_{x,\lambda}$ is linear are crucial. The proof works for the two-parameter case as well.

Part B) of the preceding theorem is useful for calculations. The first step to determine P(f, U) is to calculate T(f, U), the second is to find its U-intrinsic part. To do this we use the following criterion:

1.5 Proposition. *Let* $M \subset \mathcal{M}_{x\,\lambda}$ *be a subspace of finite codimension. Then M is U-intrinsic if and only if* $LU \,.\, M \subset M$.

Proof: See [11].

1.6 Remark. To find intrinsic parts of subspaces it is useful to note that spaces of the form

$$\mathcal{M}^k < \lambda_1, \lambda_2 >^{\ell} ,$$

where $k, \ell \in \mathbb{N}_0$ are obviously E-intrinsic and hence G-intrinsic for any subgroup G of E.

1.7 Example. Let $f = A x^3 + B x \lambda_1^2 + C \lambda_2$, where $A, B, C \neq 0$. Then

$$T(f, U) = \mathcal{M}^4 + <\lambda_1, \lambda_2>^3 + \mathbb{R}\left\{x^2\lambda_1, x^2\lambda_2, x\lambda_1\lambda_2, x\lambda_2^2, x\lambda_2, \lambda_1^2, \lambda_1\lambda_2, \lambda_2^2\right\}$$

By remark 1.6

$$\mathcal{M}^4 + <\lambda_1, \lambda_2>^3$$

is U-intrinsic and hence

$$\mathcal{M}^4 + <\lambda_1, \lambda_2>^3 \subset P(f, U)$$

holds. By applying the criterion in proposition 1.5 it follows that

$$P(f, U) = \mathcal{M}^4 + <\lambda_1, \lambda_2>^3 + \mathbb{R}\left\{x^2\lambda_2, x\lambda_1\lambda_2, x\lambda_2^2, \lambda_1\lambda_2, \lambda_2^2\right\} \quad .$$

For later use we define another concept related to intrinsic subspaces.

1.8 Definition. *Let V be a vector subspace of $\mathcal{M}_{x,\lambda}$ and G a subgroup of E. Then*

$$V^G := \sum_{v \in V} G \, . \, v \quad .$$

1.9 Proposition. *V^G is the smallest G-intrinsic subspace containing V, i. e. it satisfies the following two conditions:*

A) $V \subset V^G$ and V^G is G-intrinsic.

B) If $W \subset \mathcal{M}_{\pi,\lambda}$ is a G-intrinsic subspace containing V, then $V^G \subset W$.

The proof is straightforward.

2. Determining the U-orbits

Once P(f, U) is known, it is possible to determine the U-orbit of f, more precisely we determine

$$\frac{U \, . \, f}{P(f, U)} \, .$$

This is done by explicitly performing the coordinate changes giving U-equivalent germs to f.

2.1 Example. Let $f = A\,x^3 + B\,x\,\lambda_1^2 + C\,\lambda_2$, where $A, B, C \neq 0$. P(f, U) was

determined in the preceding section, example 1.7. Working modulo P(f, U) we obtain the U-orbit of f by first truncating equivalences $(S, X, \Lambda_1, \Lambda_2)$ in the following way:

$$\begin{aligned} X(x, \lambda_1, \lambda_2) &= x + p\,\lambda_1 + q\,\lambda_2 \\ \Lambda_1(\lambda_1, \lambda_2) &= \lambda_1 + r\,\lambda_2 \\ \Lambda_2(\lambda_1, \lambda_2) &= \lambda_2 + s\,\lambda_1^2 + t\,\lambda_1\,\lambda_2 + u\,\lambda_2^2 \, ; \end{aligned}$$

$$S(x, \lambda_1, \lambda_2) = 1 + a\,x \quad .$$

Now define

$$h(x, \lambda_1, \lambda_2) := S(x, \lambda_1, \lambda_2)\, f(X(x, \lambda_1, \lambda_2), \Lambda_1(\lambda_1, \lambda_2), \Lambda_2(\lambda_1, \lambda_2)) \, .$$

Then

$$(1 + a\,x)\left(A\,(x + p\,\lambda_1 + q\,\lambda_2)^3 + B\,(x + p\,\lambda_1 + q\,\lambda_2)(\lambda_1 + r\,\lambda_2)^2 + C\left(\lambda_2 + s\,\lambda_1^2 + t\,\lambda_1\lambda_2 + u\,\lambda_2^2\right)\right)$$

Modulo P(f, U) this reduces to

$$(1 + a\,x)\left(A\,(x + p\,\lambda_1 + q\,\lambda_2)^3 + B\,x\,\lambda_1^2 + C\left(\lambda_2 + s\,\lambda_1^2\right)\right).$$

Expanding this yields

$$C\,s\,\lambda_1^2 + 3\,A\,p\,x^2\lambda_1 + A\,x^3 + C\,\lambda_2 + C\,a\,x\,\lambda_2 + \left(B + C\,a\,s + 3\,A\,p^2\right)x\,\lambda_1^2$$

+ terms in P(f, U)

Using this result, we obtain a parametrisation of U . f / P(f, U) . The coordinates in this space are the Taylor coefficients of h. The parametrisation is

$$h = 0$$
$$h_x = 0$$
$$h_{xx} = 0$$
$$h_{\lambda_1} = 0$$
$$h_{x\lambda_1} = 0$$
$$h_{\lambda_1\lambda_1} = 2\,C\,s$$
$$h_{xx\lambda_1} = 6\,A\,p$$
$$h_{xxx} = 6\,A$$
$$h_{\lambda_2} = C$$

$$h_{x\lambda_2} = C a$$

$$h_{x\lambda_1\lambda_1} = 2B + 2Cas + 6Ap^2 .$$

According to theorem II. 5. 1 U . f / P(f, U) is an algebraic variety. Eliminating the parameters p, s and a yields the equations defining it:

$$h = 0$$
$$h_x = 0$$
$$h_{xx} = 0$$
$$h_{\lambda_1} = 0$$
$$h_{x\lambda_1} = 0$$
$$h_{\lambda_2} = C$$
$$h_{xxx} = 6A$$

$$6ACh_{x\lambda_1\lambda_1} - 6Ah_{\lambda_1\lambda_1}h_{x\lambda_2} - Ch^2_{xx\lambda_1} = 12ABC \; .$$

These are the U-recognition conditions for the germ f, i. e. each germ h whose Taylor coefficients satisfy these equations is U-equivalent to f. We rewrite the equations in the following way:

$$h = 0$$
$$h_x = 0$$
$$h_{xx} = 0$$
$$h_{\lambda_1} = 0$$
$$h_{x\lambda_1} = 0$$
$$h_{\lambda_2} = C$$
$$h_{xxx} = 6A$$

$$\begin{vmatrix} h_{xxx} & h_{xx\lambda_1} & 0 \\ h_{xx\lambda_1} & h_{x\lambda_1\lambda_1} & h_{x\lambda_2} \\ 0 & h_{\lambda_1\lambda_1} & h_{\lambda_2} \end{vmatrix} = 12\,A\,B\,C \ .$$

3. Solving the B-recognition problem

In this section we show how to obtain the B-recognition conditions of a germ, when the U-recognition conditions are already known. The following example illustrates the procedure.

3.1 Example. Let $f = \varepsilon x^3 + \delta x \lambda_1^2 + \lambda_2$, where $\varepsilon, \delta \in \{-1, +1\}$. Since

$$B = TU = UT$$

$h \in B \,.\, f$ holds if and only if $h \in U \,.\, k$ for some $k \in T \,.\, f$. The T-orbit of f is

$$T \,.\, f = \left\{ \varepsilon \mu \nu^3 x^3 + \delta \mu \nu m^2 x \lambda_1^2 + \mu n \lambda_2 \mid \mu, \nu > 0; m, n \neq 0 \right\} .$$

This expression shows that $k \in T \,.\, f$ holds, if and only if k is of the form

$$A x^3 + B x \lambda_1^2 + C \lambda_2 ,$$

where A, B and C satisfy the following conditions:

$$\begin{aligned} \operatorname{sg} A &= \varepsilon \\ \operatorname{sg} B &= \delta \\ C &\neq 0 . \end{aligned} \tag{3.1}$$

The conditions for $h \in U \,.\, k$ were derived in example 2. 1 in the previous section. Combining these with (3.1), it follows that the B-recognition conditions for f are

$$h = 0$$
$$h_x = 0$$
$$h_{xx} = 0$$
$$\operatorname{sg} h_{xxx} = \varepsilon$$
$$h_{\lambda_1} = 0$$
$$h_{\lambda_2} \neq 0$$
$$h_{x\lambda_1} = 0$$

$$\begin{vmatrix} h_{xxx} & h_{xx\lambda_1} & 0 \\ h_{xx\lambda_1} & h_{x\lambda_1\lambda_1} & h_{x\lambda_2} \\ 0 & h_{\lambda_1\lambda_1} & h_{\lambda_2} \end{vmatrix} \neq 0 \ .$$

We now give a list of certain germs and the corresponding B-recognition conditions. The germs have been chosen according to the following consideration: A K-versal unfolding of the germ x^m $(m \geq 2)$ is given by

$$x^m + \alpha_1 x^{m-2} + \ldots + \alpha_{m-2} x + \alpha_{m-1} \ ,$$

where $\alpha_1, \ldots, \alpha_{m-1} \in \mathbb{R}$ (See [1], [4], [5] and [10].). Hence every two-parameter germ $f(x, \lambda_2, \lambda_1)$ is E-equivalent to some germ of the form

$$x^m + \varphi_1(\lambda_1, \lambda_2) x^{m-2} + \ldots + \varphi_{m-2}(\lambda_1, \lambda_2) x + \varphi_{m-1}(\lambda_1, \lambda_2) \ ,$$

where $\varphi_1, \ldots, \varphi_{m-1} \in \mathcal{E}_\lambda$. To obtain germs of low codimension we consider $m = 2, 3, 4$ and choose the germs φ_i to be linear or quadratic in λ_1 and λ_2 .

$$\varepsilon x^2 + \lambda_1$$

where $\varepsilon \in \{-1, +1\}$.

$$h = 0$$
$$h_x = 0$$
$$\operatorname{sg} h_{xx} = \varepsilon$$
$$h_{\lambda_1} \neq 0$$

Table 3.1

$$\varepsilon x^2 + \lambda_2$$

where $\varepsilon \in \{-1, +1\}$

$$h = 0$$
$$h_x = 0$$
$$\operatorname{sg} h_{xx} = \varepsilon$$
$$h_{\lambda_1} = 0$$
$$h_{\lambda_2} \neq 0$$

Table 3.2

$$\varepsilon x^3 + x\lambda_1 + \lambda_2$$

where $\varepsilon \in \{ -1, +1 \}$

$$h = 0$$

$$h_x = 0$$

$$h_{xx} = 0$$

$$\mathrm{sg}\, h_{xxx} = \varepsilon$$

$$h_{\lambda_1} = 0$$

$$\begin{vmatrix} h_{\lambda_1} & h_{\lambda_2} \\ h_{x\lambda_1} & h_{x\lambda_2} \end{vmatrix} \neq 0$$

Table 3.3

$$\varepsilon x^3 + x\lambda_2 + \lambda_1$$

where $\varepsilon \in \{ -1, +1 \}$

$$h = 0$$

$$h_x = 0$$

$$h_{xx} = 0$$

$$\operatorname{sg} h_{xxx} = \varepsilon$$

$$h_{\lambda_1} \neq 0$$

$$\begin{vmatrix} h_{\lambda_1} & h_{\lambda_2} \\ h_{x\lambda_1} & h_{x\lambda_2} \end{vmatrix} \neq 0$$

Table 3.4

$$\varepsilon x^2 + \delta(\lambda_1^2 + \lambda_2^2)$$

where $\varepsilon, \delta \in \{-1, +1\}$

$$h = 0$$
$$h_x = 0$$
$$h_{\lambda_1} = 0$$
$$h_{\lambda_2} = 0$$
$$\mathrm{sg}\, h_{xx} = \varepsilon$$
$$\mathrm{sg}\, D_1 = \varepsilon\,\delta$$
$$\mathrm{sg}\, H = \varepsilon\ ,$$

where

$$D_1 = \begin{vmatrix} h_{xx} & h_{x\lambda_1} \\ h_{x\lambda_1} & h_{\lambda_1\lambda_1} \end{vmatrix}$$

and

$$H = \begin{vmatrix} h_{xx} & h_{x\lambda_1} & h_{x\lambda_2} \\ h_{x\lambda_1} & h_{\lambda_1\lambda_1} & h_{\lambda_1\lambda_2} \\ h_{x\lambda_2} & h_{\lambda_1\lambda_2} & h_{\lambda_2\lambda_2} \end{vmatrix} .$$

Table 3.5

$$\varepsilon x^2 + \delta(\lambda_1^2 - \lambda_2^2)$$

where $\varepsilon, \delta \in \{-1, +1\}$

$$h = 0$$
$$h_x = 0$$
$$h_{\lambda_1} = 0$$
$$h_{\lambda_2} = 0$$
$$\mathrm{sg}\, h_{xx} = \varepsilon$$
$$\mathrm{sg}\, D_1 = \varepsilon\,\delta$$
$$\mathrm{sg}\, H = -\varepsilon,$$

where

$$D_1 = \begin{vmatrix} h_{xx} & h_{x\lambda_1} \\ h_{x\lambda_1} & h_{\lambda_1\lambda_1} \end{vmatrix}$$

and

$$H = \begin{vmatrix} h_{xx} & h_{x\lambda_1} & h_{x\lambda_2} \\ h_{x\lambda_1} & h_{\lambda_1\lambda_1} & h_{\lambda_1\lambda_2} \\ h_{x\lambda_2} & h_{\lambda_1\lambda_2} & h_{\lambda_2\lambda_2} \end{vmatrix}.$$

Table 3.6

$$\varepsilon x^3 + \delta x \lambda_1^2 + \lambda_2$$

where $\varepsilon, \delta \in \{ -1, +1 \}$

$$h = 0$$
$$h_x = 0$$
$$h_{xx} = 0$$
$$\text{sg } h_{xxx} = \varepsilon$$
$$h_{\lambda_1} = 0$$
$$h_{\lambda_2} \neq 0$$
$$h_{x\lambda_1} = 0$$

$$\begin{vmatrix} h_{xxx} & h_{xx\lambda_1} & 0 \\ h_{xx\lambda_1} & h_{x\lambda_1\lambda_1} & h_{x\lambda_2} \\ 0 & h_{\lambda_1\lambda_1} & h_{\lambda_2} \end{vmatrix} \neq 0$$

Table 3.7

$$\varepsilon x^4 + x\lambda_1 + \lambda_2$$

where $\varepsilon \in \{-1, +1\}$

$$h = 0$$

$$h_x = 0$$

$$h_{xx} = 0$$

$$h_{xxx} = 0$$

$$\text{sg}\, h_{xxxx} = \varepsilon$$

$$h_{\lambda_1} = 0$$

$$\Delta \neq 0$$

where

$$\Delta = \begin{vmatrix} h_{\lambda_1} & h_{\lambda_2} \\ h_{x\lambda_1} & h_{x\lambda_2} \end{vmatrix}$$

Table 3.8

4. Solving the E-recognition problem

Once the B-recognition problem has been solved, there is one additional step to solve the E-recognition problem. This procedure is based on proposition II. 2. 2 and is therefore a consequence of the Bruhat decomposition for GL(2, $\mathbb{R}$). Since this decomposition is valid for GL(n, $\mathbb{R}$), the method described below can be applied to bifurcations having more than two parameters.

4.1 Proposition. *Let f and h be germs in* $\mathcal{M}_{x,\lambda}$. *Then the following statements are equivalent:*

A) $h \in E \,.\, f$.

B) Either $h \in B \,.\, f$ *or there exists* $\sigma \in \mathbb{R}$ *such that* $h(x, \sigma\lambda_1 + \lambda_2, \lambda_1) \in B \,.\, f$.

Proof: We use the decomposition of E given in proposition II. 2. 2 .

A) $\Rightarrow$ B): Let $h = e \,.\, f$, where $e \in E$. According to remark II. 2. 4 and proposition II. 2. 2 either $e \in B$ or $e = n\, w\, b$, where $n \in N$, $w \in W$ and $b \in B$. In the latter case it follows that

$$(w^{-1} n^{-1}) \,.\, h = b \,.\, f \quad .$$

By the definition of the groups N and W there exists a $\sigma \in \mathbb{R}$ such that

$$\Big(\big(w^{-1} n^{-1} \big) \,.\, h \Big)(x, \lambda_1, \lambda_2) = h(x, \sigma\lambda_1 + \lambda_2, \lambda_1)$$

holds. Hence $h(x, \sigma\lambda_1 + \lambda_2, \lambda_1) \in B \,.\, f$.

The implication B) $\Rightarrow$ A) is proved similarly. □

In order to apply part B) of the proposition it is necessary to know how the Taylor coefficients of $h(x, \sigma\lambda_1 + \lambda_2, \lambda_1)$ relate to those of h. This relationship is as follows:

4.2 Proposition. *Let h be a germ in* $\mathcal{E}_{x,\lambda}$ *and*

$$h^*(x, \lambda_1, \lambda_2) := h(x, \sigma\lambda_1 + \lambda_2, \lambda_1)$$

for some $\sigma \in \mathbb{R}$ *. Then*

$$h^*_{x^\alpha \lambda_1^\beta \lambda_2^\gamma} = \sum_{k=0}^{\beta} \binom{\beta}{k} \sigma^k h_{x^\alpha \lambda_1^{\gamma+k} \lambda_2^{\beta-k}} \quad .$$

Proof: Fix $(\alpha', \beta', \gamma') \in \mathbb{N}_0^3$, choose an integer m such that $m \geq \alpha' + \beta' + \gamma'$ and consider the m-jet of h:

$$j^m h(x, \lambda_1, \lambda_2) = \sum_{\substack{(\alpha,\beta,\gamma)\in\mathbb{N}_0^3 \\ \alpha+\beta+\gamma\leq m}} \frac{1}{\alpha!\beta!\gamma!} h_{x^\alpha \lambda_1^\beta \lambda_2^\gamma} x^\alpha \lambda_1^\beta \lambda_2^\gamma \quad .$$

This implies

$$j^m h^*(x, \lambda_1, \lambda_2) = \sum_{\substack{(\alpha,\beta,\gamma)\in\mathbb{N}_0^3 \\ \alpha+\beta+\gamma\leq m}} \frac{1}{\alpha!\beta!\gamma!} h_{x^\alpha \lambda_1^\beta \lambda_2^\gamma} \sum_{k=0}^{\beta} \binom{\beta}{k} \sigma^k x^\alpha \lambda_1^{\gamma+k} \lambda_2^{\beta-k} \quad .$$

The coefficient of $x^{\alpha'} \lambda_1^{\beta'} \lambda_2^{\gamma'}$ in this expression is

$$\sum_{k=0}^{\beta'} \frac{1}{\alpha'!(\beta'-k)!k!\gamma'!}\,\sigma^k h_{x^{\alpha'}\lambda_1^{\gamma'+k}\lambda_2^{\beta'-k}}\,.$$

To obtain $h^{*}_{x^{\alpha'}\lambda_1^{\beta'}\lambda_2^{\gamma}}$ we multiply by $\alpha'!\,\beta'!\,\gamma'!$:

$$h^{*}_{x^{\alpha'}\lambda_1^{\beta'}\lambda_2^{\gamma}} = \sum_{k=0}^{\beta'} \binom{\beta'}{k}\sigma^k h_{x^{\alpha'}\lambda_1^{\gamma+k}\lambda_2^{\beta'-k}}\,. \;\square$$

We now solve the E-recognition problem for three particular germs. These results are stated in theorems 4.3, 4.4 and 4.6.

4.3 Theorem. *Let $f = \varepsilon x^2 + \delta(\lambda_1^2 + \lambda_2^2)$, where $\varepsilon, \delta \in \{-1, +1\}$ and let h be a germ in $\mathcal{M}_{x\lambda}$. Then h is E-equivalent to f if and only if h satisfies the following conditions:*

$$h = 0 \tag{4.1}$$
$$h_x = 0 \tag{4.2}$$
$$sg\, h_{xx} = \varepsilon \tag{4.3}$$
$$h_{\lambda_1} = 0 \tag{4.4}$$
$$h_{\lambda_2} = 0 \tag{4.5}$$
$$sg\, D_1 = \varepsilon\,\delta \tag{4.6}$$
$$sg\, H = \varepsilon\,, \tag{4.7}$$

where

$$D_1 = \begin{vmatrix} h_{xx} & h_{x\lambda_1} \\ h_{x\lambda_1} & h_{\lambda_1\lambda_1} \end{vmatrix} .$$

and

$$H = \begin{vmatrix} h_{xx} & h_{x\lambda_1} & h_{x\lambda_2} \\ h_{x\lambda_1} & h_{\lambda_1\lambda_1} & h_{\lambda_1\lambda_2} \\ h_{x\lambda_2} & h_{\lambda_1\lambda_2} & h_{\lambda_2\lambda_2} \end{vmatrix} .$$

Proofs both of this and the next theorem will be given following remark 4.5 .

4.4 Theorem. *Let* $f = \varepsilon x^2 + \lambda_1^2 - \lambda_2^2$ *, where* $\varepsilon \in \{-1, +1\}$ *and let h be a germ in* $\mathcal{M}_{x,\lambda}$ *. Then h is E-equivalent to f if and only if h satisfies the following conditions:*

$$h = 0 \tag{4.8}$$

$$h_x = 0 \tag{4.9}$$

$$sg\, h_{xx} = \varepsilon \tag{4.10}$$

$$h_{\lambda_1} = 0 \tag{4.11}$$

$$h_{\lambda_2} = 0 \tag{4.12}$$

$$sg\, H = -\varepsilon , \tag{4.13}$$

where

$$H = \begin{vmatrix} h_{xx} & h_{x\lambda_1} & h_{x\lambda_2} \\ h_{x\lambda_1} & h_{\lambda_1\lambda_1} & h_{\lambda_1\lambda_2} \\ h_{x\lambda_2} & h_{\lambda_1\lambda_2} & h_{\lambda_2\lambda_2} \end{vmatrix} .$$

4.5 Remark. The determinant H appearing in theorems 4.3 and 4.4 is the determinant of the Hessian of the function h . The results show that no third-order-terms appear in the recognition conditions. That is, f is 2-determined. Therefore the classification corresponds here to the classification of quadratic forms allowing linear coordinate changes which preserve the sign of h_{xx} . For example, taking $\varepsilon = \delta = 1$ in theorem 4.3 the recognition conditions

$$h_{xx} > 0$$
$$D_1 > 0$$
$$H > 0$$

are exactly the conditions for the quadratic form defined by the symmetric matrix

$$\begin{pmatrix} h_{xx} & h_{x\lambda_1} & h_{x\lambda_2} \\ h_{x\lambda_1} & h_{\lambda_1\lambda_1} & h_{\lambda_1\lambda_2} \\ h_{x\lambda_2} & h_{\lambda_1\lambda_2} & h_{\lambda_2\lambda_2} \end{pmatrix}$$

to be positive definite.

Proof of theorem 4.3: We apply proposition 4.1. Conditions (4.1) — (4.7) are identical with the conditions that $h \in B . f$ (Compare table 3.5.). Hence the aim is to show that (4.1) — (4.7) are invariant under the transformation of Taylor coefficients given in proposition 4.2. For $\sigma \in \mathbb{R}$ let $h^*(x, \lambda_1, \lambda_2) := h(x, \sigma \lambda_1 + \lambda_2, \lambda_1)$. The relevant Taylor coefficients of h^* are given by

$$
\begin{aligned}
h^* &= h \\
h^*_x &= h_x \\
h^*_{xx} &= h_{xx} \\
h^*_{\lambda_1} &= h_{\lambda_2} + \sigma h_{\lambda_1} \\
h^*_{\lambda_2} &= h_{\lambda_1} \\
h^*_{x\lambda_1} &= h_{x\lambda_2} + \sigma h_{x\lambda_1} \\
h^*_{x\lambda_2} &= h_{x\lambda_1} \\
h^*_{\lambda_1\lambda_1} &= h_{\lambda_2\lambda_2} + 2\sigma h_{\lambda_1\lambda_2} + \sigma^2 h_{\lambda_1\lambda_1} \\
h^*_{\lambda_1\lambda_2} &= h_{\lambda_1\lambda_2} + \sigma h_{\lambda_1\lambda_1} \\
h^*_{\lambda_2\lambda_2} &= h_{\lambda_1\lambda_1} \ .
\end{aligned}
$$

Applying the B-recognition conditions to h^* and substituting these expressions yields the following: (4.1), (4.2) and (4.3) are obviously preserved. (4.4) and (4.5) are transformed into

$$
\begin{aligned}
h_{\lambda_2} + \sigma h_{\lambda_1} &= 0 \\
h_{\lambda_1} &= 0 \ ,
\end{aligned}
$$

and these equations are equivalent to

$$
\begin{aligned}
h_{\lambda_1} &= 0 \\
h_{\lambda_2} &= 0 \ .
\end{aligned}
$$

Hence (4.4) and (4.5) are preserved.

Now consider conditions (4.6) and (4.7). Let $\Psi := h_{xx} H$. By (4.3) condition (4.7) is equivalent to

$$\Psi > 0 . \tag{4.14}$$

The following identity holds:

$$\Psi = \begin{vmatrix} D_1 & D^* \\ D^* & D_2 \end{vmatrix} , \tag{4.15}$$

where

$$D^* = \begin{vmatrix} h_{xx} & h_{x\lambda_1} \\ h_{x\lambda_2} & h_{\lambda_1\lambda_2} \end{vmatrix}$$

and

$$D_2 = \begin{vmatrix} h_{xx} & h_{x\lambda_2} \\ h_{x\lambda_2} & h_{\lambda_2\lambda_2} \end{vmatrix} .$$

Now we determine the transforms of the three determinants D_1, D^* and D_2 . After some calculations it turns out that D_1 transforms into $D_2 + 2\,\sigma D^* + \sigma^2 D_1$, D^* into $D^* + \sigma D_1$ and D_2 into D_1. Now consider conditions (4.6) and (4.14). (4.14) implies

$$\mathrm{sg}\, D_1 = \mathrm{sg}\, D_2 ,$$

since

$$D_1 D_2 > \left(D^*\right)^2 \geq 0 .$$

Hence (4.6) is equivalent to sg $D_2 = \varepsilon\,\delta$. This transforms into sg $D_1 = \varepsilon\,\delta$. Hence condition (4.6) is preserved. The transform of the determinant in (4.15) is

$$\begin{vmatrix} D_2 + 2\sigma D^* + \sigma^2 D_1 & D^* + \sigma D_1 \\ D^* + \sigma D_1 & D_1 \end{vmatrix}$$

$$= \quad D_1 D_2 + 2\sigma D_1 D^* + \sigma^2 D_1^2 - \left(D^*\right)^2 - 2\sigma D_1 D^* - \sigma^2 D_1^2$$

$$= \begin{vmatrix} D_1 & D^* \\ D^* & D_2 \end{vmatrix} .$$

Hence Ψ is invariant under the transformation and condition (4.14) is preserved as well. Since h_{xx} is invariant under the transformation, it follows that H is invariant. This proves the result. □

Proof of theorem 4.4: According to table (3.6) the B-recognition conditions for f are (4.8) — (4.13) plus

$$\mathrm{sg}\, D_1 = \varepsilon \ . \tag{4.16}$$

The invariance of (4.13) under the transformation of the Taylor coefficients follows in the same way as in the preceding proof. Using the same definition for Ψ as above (4.13) is equivalent to

$$\Psi < 0 \ . \tag{4.17}$$

Now consider condition (4.16), which transforms into

$$\mathrm{sg}\,(D_2 + 2\,\sigma D^* + \sigma^2 D_1) = \varepsilon \ . \tag{4.18}$$

Suppose that $D_1 \neq 0$. Then the quadratic polynomial in (4.18) has $-4\,\Psi$ as its discriminant. By (4.17) this discriminant is positive. Hence the polynomial assumes negative and positive values, since it has two distinct real roots. Suppose now that $D_1 = 0$. By (4.17) D^* does not vanish and hence the expression in (4.18) assumes positive and negative values.

We have shown that in both cases there exist values of σ such that (4.18) holds without further restrictions on D_1, D^* and D_2 . By proposition 4.1 the result follows. □

4.6 Theorem. *Let* $f = \varepsilon x^3 + \delta x \lambda_1^2 + \lambda_2$ *, where* $\varepsilon, \delta \in \{-1, +1\}$ *and let* h *be a germ in* $\mathcal{M}_{x\lambda}$ *. Then* h *is E-equivalent to* f *if and only if* h *satisfies the following conditions:*

$$h = 0 \tag{4.19}$$

$$h_x = 0 \tag{4.20}$$

$$h_{xx} = 0 \tag{4.21}$$

$$sg\, h_{xxx} = \varepsilon \tag{4.22}$$

$$\Delta = 0 \tag{4.23}$$

$$\Gamma \neq 0 \tag{4.24}$$

where

$$\Delta = \begin{vmatrix} h_{\lambda_1} & h_{\lambda_2} \\ h_{x\lambda_1} & h_{x\lambda_2} \end{vmatrix} ,$$

$$\Gamma = \begin{vmatrix} K_1 & 2K^* - K_2 \\ h_{\lambda_1} & h_{\lambda_2} \end{vmatrix}$$

and where

$$K_1 = \begin{vmatrix} h_{xxx} & h_{xx\lambda_1} & 0 \\ h_{xx\lambda_1} & h_{x\lambda_1\lambda_1} & h_{x\lambda_2} \\ 0 & h_{\lambda_1\lambda_1} & h_{\lambda_2} \end{vmatrix} ,$$

$$K^* = \begin{vmatrix} h_{xxx} & h_{xx\lambda_2} & 0 \\ h_{xx\lambda_1} & h_{x\lambda_1\lambda_2} & h_{x\lambda_2} \\ 0 & h_{\lambda_1\lambda_2} & h_{\lambda_2} \end{vmatrix}$$

and

$$K_2 = \begin{vmatrix} h_{xxx} & h_{xx\lambda_2} & 0 \\ h_{xx\lambda_2} & h_{x\lambda_2\lambda_2} & h_{x\lambda_1} \\ 0 & h_{\lambda_2\lambda_2} & h_{\lambda_1} \end{vmatrix} .$$

Proof: The proof is divided into two steps.

Step 1: We show that the B-recognition conditions for f given in table 3.7 are equivalent to (4.19) — (4.24) plus the condition $h_{\lambda_1} = 0$. It is sufficient to show that

$$h_{\lambda_1} = 0$$

$$h_{\lambda_2} \neq 0$$
$$h_{x\lambda_1} = 0$$
$$K_1 \neq 0$$

are equivalent to

$$h_{\lambda_1} = 0 \qquad (4.25)$$
$$\Delta = 0$$
$$\Gamma \neq 0 \quad .$$

Assume the conditions stated first hold. Since $h_{\lambda_1} = 0$, $\Gamma = h_{\lambda_2} K_1$. Since $h_{\lambda_2} \neq 0$, it follows that $\Gamma \neq 0$. $h_{\lambda_1} = h_{x\lambda_1} = 0$ implies $\Delta = 0$.
To show the converse, note that again $\Gamma = h_{\lambda_2} K_1$. Hence $h_{\lambda_2} \neq 0$ and $K_1 \neq 0$. Since $0 = \Delta = - h_{\lambda_2} h_{x\lambda_1}$, it follows that $h_{x\lambda_1} = 0$.

Step 2: We apply conditions (4.19) — (4.24) and (4.25) to the function

$$h^*(x, \lambda_1, \lambda_2) := h(x, \sigma \lambda_1 + \lambda_2, \lambda_1).$$

We express the Taylor coefficients of h^* according to proposition 4.2. Apart from the formulae stated in the proof of theorem 4.3 we need

$$\begin{aligned}
h^*_{xx\lambda_1} &= h_{xx\lambda_2} + \sigma h_{xx\lambda_1} \\
h^*_{xx\lambda_2} &= h_{xx\lambda_1} \\
h^*_{x\lambda_1\lambda_1} &= h_{x\lambda_2\lambda_2} + 2\sigma h_{x\lambda_1\lambda_2} + \sigma^2 h_{x\lambda_1\lambda_1} \\
h^*_{x\lambda_1\lambda_2} &= h_{x\lambda_1\lambda_2} + \sigma h_{x\lambda_1\lambda_1} \\
h^*_{x\lambda_2\lambda_2} &= h_{x\lambda_1\lambda_1} \quad .
\end{aligned}$$

Condition (4.25) transforms into

$$h_{\lambda_2} + \sigma h_{\lambda_1} = 0 \ . \tag{4.26}$$

The transform of Δ is

$$\begin{vmatrix} h_{\lambda_2} + \sigma h_{\lambda_1} & h_{\lambda_1} \\ h_{x\lambda_2} + \sigma h_{x\lambda_1} & h_{x\lambda_1} \end{vmatrix}$$

$$= \begin{vmatrix} h_{\lambda_2} & h_{\lambda_1} \\ h_{x\lambda_2} & h_{x\lambda_1} \end{vmatrix} + \sigma \begin{vmatrix} h_{\lambda_1} & h_{\lambda_1} \\ h_{x\lambda_1} & h_{x\lambda_1} \end{vmatrix}$$

$$= -\Delta \ .$$

Hence (4.23) is preserved. Now consider the transform of Γ. After some calculation it turns out that K_1 transforms into $K_2 - 2K^* - \sigma K_1$. Let Q denote the transform of $2K^* - K_2$. Then Γ transforms into

$$\begin{vmatrix} K_2 - 2K^* - \sigma K_1 & Q \\ h_{\lambda_2} + \sigma h_{\lambda_1} & h_{\lambda_1} \end{vmatrix}$$

Condition (4.26) implies that this is equal to

$$h_{\lambda_1}(K_2 - 2K^* - \sigma K_1)$$

$$= h_{\lambda_1}(K_2 - 2K^*) - \sigma h_{\lambda_1} K_1$$

By (4.26) this is equal to

$$h_{\lambda_1}(K_2 - 2K^*) + h_{\lambda_2} K_1$$

$$= \begin{vmatrix} K_1 & 2K^* - K_2 \\ h_{\lambda_1} & h_{\lambda_2} \end{vmatrix}$$

$$= \Gamma \ .$$

This calculation shows that Γ is invariant under the transformation and hence condition (4.24) is preserved. By (4.24) h_{λ_1} and h_{λ_2} cannot both vanish. Therefore there exists a $\sigma \in \mathbb{R}$ satisfying (4.26) if and only if $h_{\lambda_1} \neq 0$. It is trivial to show that (4.19) — (4.22) are preserved.

We have shown that $h^* \in B$. f holds if and only if (4.19) — (4.22) hold and

$$h_{\lambda_1} \neq 0$$
$$\Delta = 0$$
$$\Gamma \neq 0 \ .$$

By proposition 4.1 the result in the theorem follows. □

5. Data for E-equivalence

We now give lists of the E-recognition conditions for a collection of normal forms. These germs correspond to those in section 3, except that some of them are E-equivalent to each other like

$$x^2 + \lambda_1 \text{ and } x^2 + \lambda_2 .$$

The reasons for choosing the germs are discussed in section 3.

Proofs for the recognition conditions in tables 5.3, 5.4 and 5.5 are given in section 4. The proofs for the other results are considerably easier. In fact, the relevant steps appear in the proofs in section 4 as well — as the rather trivial parts. For this reason these proofs are omitted here.

As additional information we include the codimension and — provided this is positive — the unfolding terms are given. Details concerning how to calculate $\mathcal{E}_{x,\lambda}/T_e(f, E)$ can be found in example II. 3. 6 for two cases. The calculations for the other cases are straightforward and similar to example II. 3. 6. 1 .

$$\varepsilon x^2 + \lambda_1$$

where $\varepsilon \in \{-1, +1\}$.

codimension 0

$$h = 0$$
$$h_x = 0$$
$$\mathrm{sg}\, h_{xx} = \varepsilon$$
$$(h_{\lambda_1}, h_{\lambda_2}) \neq (0, 0)$$

Table 5.1

$$\varepsilon x^3 + x\lambda_1 + \lambda_2$$

where $\varepsilon \in \{-1, +1\}$

codimension 0

$$h = 0$$
$$h_x = 0$$
$$h_{xx} = 0$$
$$\mathrm{sg}\, h_{xxx} = \varepsilon$$
$$\Delta \neq 0$$

where

$$\Delta = \begin{vmatrix} h_{\lambda_1} & h_{\lambda_2} \\ h_{x\lambda_1} & h_{x\lambda_2} \end{vmatrix}$$

Table 5.2

$$\varepsilon x^2 + \delta(\lambda_1^2 + \lambda_2^2)$$

where $\varepsilon, \delta \in \{-1, +1\}$

codimension 1

unfolding term: 1

$$h = 0$$
$$h_x = 0$$
$$\mathrm{sg}\, h_{xx} = \varepsilon$$
$$h_{\lambda_1} = 0$$
$$h_{\lambda_2} = 0$$
$$\mathrm{sg}\, D_1 = \varepsilon\,\delta$$
$$\mathrm{sg}\, H = \varepsilon\,,$$

where

$$D_1 = \begin{vmatrix} h_{xx} & h_{x\lambda_1} \\ h_{x\lambda_1} & h_{\lambda_1\lambda_1} \end{vmatrix}$$

and

$$H = \begin{vmatrix} h_{xx} & h_{x\lambda_1} & h_{x\lambda_2} \\ h_{x\lambda_1} & h_{\lambda_1\lambda_1} & h_{\lambda_1\lambda_2} \\ h_{x\lambda_2} & h_{\lambda_1\lambda_2} & h_{\lambda_2\lambda_2} \end{vmatrix} .$$

Table 5.3

$$\varepsilon x^2 + \lambda_1^2 - \lambda_2^2$$

where $\varepsilon \in \{-1, +1\}$

codimension 1

unfolding term: 1

$$h = 0$$
$$h_x = 0$$
$$\operatorname{sg} h_{xx} = \varepsilon$$
$$h_{\lambda_1} = 0$$
$$h_{\lambda_2} = 0$$
$$\operatorname{sg} H = -\varepsilon ,$$

where

$$H = \begin{vmatrix} h_{xx} & h_{x\lambda_1} & h_{x\lambda_2} \\ h_{x\lambda_1} & h_{\lambda_1\lambda_1} & h_{\lambda_1\lambda_2} \\ h_{x\lambda_2} & h_{\lambda_1\lambda_2} & h_{\lambda_2\lambda_2} \end{vmatrix} .$$

Table 5.4

$$\varepsilon x^3 + \delta x \lambda_1^2 + \lambda_2$$

where $\varepsilon, \delta \in \{ -1, +1 \}$

codimension 1

unfolding term: x

$$h = 0$$
$$h_x = 0$$
$$h_{xx} = 0$$
$$sg\, h_{xxx} = \varepsilon$$
$$\Delta = 0$$
$$\Gamma \neq 0$$

where

$$\Delta = \begin{vmatrix} h_{\lambda_1} & h_{\lambda_2} \\ h_{x\lambda_1} & h_{x\lambda_2} \end{vmatrix} ,$$

$$\Gamma = \begin{vmatrix} K_1 & 2K^* - K_2 \\ h_{\lambda_1} & h_{\lambda_2} \end{vmatrix}$$

and where

$$K_1 = \begin{vmatrix} h_{xxx} & h_{xx\lambda_1} & 0 \\ h_{xx\lambda_1} & h_{x\lambda_1\lambda_1} & h_{x\lambda_2} \\ 0 & h_{\lambda_1\lambda_1} & h_{\lambda_2} \end{vmatrix} ,$$

$$K^{*} = \begin{vmatrix} h_{xxx} & h_{xx\lambda_2} & 0 \\ h_{xx\lambda_1} & h_{x\lambda_1\lambda_2} & h_{x\lambda_2} \\ 0 & h_{\lambda_1\lambda_2} & h_{\lambda_2} \end{vmatrix}$$

and

$$K_2 = \begin{vmatrix} h_{xxx} & h_{xx\lambda_2} & 0 \\ h_{xx\lambda_2} & h_{x\lambda_2\lambda_2} & h_{x\lambda_1} \\ 0 & h_{\lambda_2\lambda_2} & h_{\lambda_1} \end{vmatrix} .$$

Table 5.5

$$\varepsilon x^4 + x\lambda_1 + \lambda_2$$

where $\varepsilon \in \{-1, +1\}$

codimension 1

unfolding term: x^2

$$h = 0$$
$$h_x = 0$$
$$h_{xx} = 0$$
$$h_{xxx} = 0$$
$$\mathrm{sg}\, h_{xxxx} = \varepsilon$$
$$\Delta \neq 0$$

where

$$\Delta = \begin{vmatrix} h_{\lambda_1} & h_{\lambda_2} \\ h_{x\lambda_1} & h_{x\lambda_2} \end{vmatrix}$$

Table 5.6

6. The classification theorem

In this section we give the classification of two-parameter bifurcations up to codimension one.

6.1 Theorem. *Let h be a germ in* $\mathcal{E}_{x\lambda}$ *satisfying* $h = h_x = 0$. *Let the codimension of h be less than or equal to one. Then h is E-equivalent to one of the following germs:*

$$\varepsilon x^2 + \lambda_1$$

$$\varepsilon x^2 + \lambda_1^2 - \lambda_2^2$$

$$\varepsilon x^2 + \delta(\lambda_1^2 + \lambda_2^2)$$

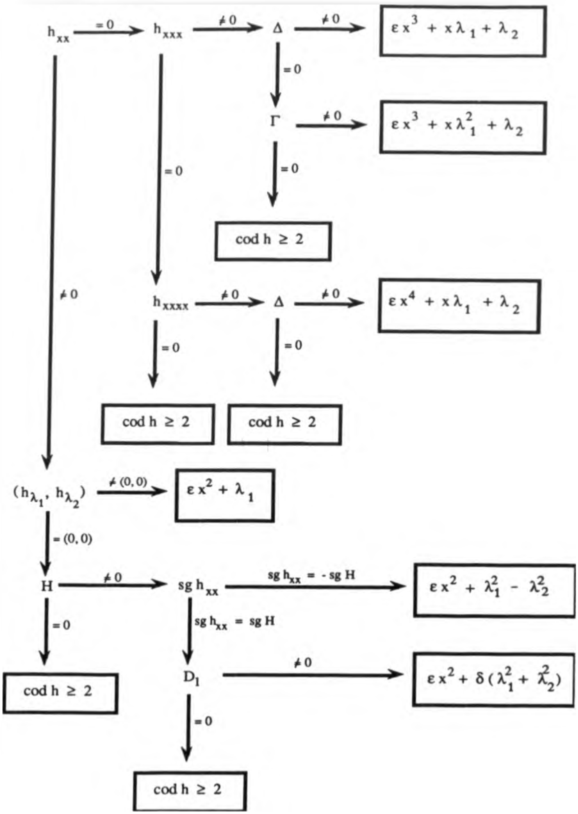
$$\varepsilon x^3 + x\lambda_1 + \lambda_2$$

$$\varepsilon x^3 + \delta x\lambda_1^2 + \lambda_2$$

$$\varepsilon x^4 + x\lambda_1 + \lambda_2 \, ,$$

where $\varepsilon, \delta \in \{-1, +1\}$.

Proof: The essential part of the proof consists of inspecting the E-recognition conditions for the normal forms given in section 5. The following diagram makes the procedure more transparent.



h_{xx}
$= 0$
h_{xxx}
$\neq 0$
Δ
$\neq 0$
$\varepsilon x^3 + x\lambda_1 + \lambda_2$
$= 0$
Γ
$\neq 0$
$\varepsilon x^3 + x\lambda_1^2 + \lambda_2$
$= 0$
$= 0$
cod h $\geq$ 2
$\neq 0$
h_{xxxx}
$\neq 0$
Δ
$\neq 0$
$\varepsilon x^4 + x\lambda_1 + \lambda_2$
$= 0$
$= 0$
cod h $\geq$ 2
cod h $\geq$ 2
$(h_{\lambda_1}, h_{\lambda_2})$
$\neq (0,0)$
$\varepsilon x^2 + \lambda_1$
$= (0,0)$
H
$\neq 0$
sg h_{xx}
sg h_{xx} = - sg H
$\varepsilon x^2 + \lambda_1^2 - \lambda_2^2$
$= 0$
sg h_{xx} = sg H
cod h $\geq$ 2
D_1
$\neq 0$
$\varepsilon x^2 + \delta(\lambda_1^2 + \lambda_2^2)$
$= 0$
cod h $\geq$ 2

Fig. 6.1

Suppose $h \in \mathcal{E}_{x\lambda}$ satisfies $h = h_x = 0$. Starting with h_{xx} and following the arrows in the flow chart, the diagram shows how the Taylor coefficients determine the equivalence class of h.

There are five paths in the flow chart which terminate with the statement cod $h \geq 2$. This follows from the fact that for each of these h satisfies five defining conditions (including $h = h_x = 0$) — these are denoted by arrows marked by " $= 0$ ". The codimension of a germ equals the number of independent defining conditions minus three — provided this is a non-negative number. The proof of this statement is analogous to the corresponding one for one-parameter bifurcations (See [7], corollary III. 2. 6., p. 126.). □

CHAPTER IV

1. Efficient calculation of the higher-order terms

In this section we describe a result, which is relevant for choosing the normal forms which were used in section III. 2 to calculate the U-orbits. The following example illustrates the importance of choosing the normal form appropriately.

1.1 Example. Consider the germ $f = \varepsilon x^3 + \delta x \lambda_1^2 + \lambda_2$, where $\varepsilon, \delta \in \{-1, +1\}$. f is E-equivalent to $g = \varepsilon x^3 + \delta x \lambda_2^2 + \lambda_1$. It is possible to solve the E-recognition problem for g instead of f. However, the calculations are a great deal more complicated for the following reason: The higher-order terms for f are given by

$$P(f, U) = \mathcal{M}^4 + <\lambda_1, \lambda_2>^3 + \mathbb{R}\left\{ x^2\lambda_2, x\lambda_1\lambda_2, x\lambda_2^2, \lambda_1\lambda_2, \lambda_2^2 \right\} .$$

(Compare example III. 2. 1.), whereas

$$P(g, U) = \mathcal{M}^4 + <\lambda_1, \lambda_2>^3 .$$

Hence $U \, . \, g \, / \, P(g, U)$ has five extra dimensions compared to $U \, . \, f \, / \, P(f, U)$. As a consequence the parametrisation of $U \, . \, g \, / \, P(g, U)$ turns out to be very complicated. However, it is possible to check that it eventually results in the same E-recognition conditions as for f .[1] Clearly, it is advantageous to choose f and not g as the normal form.

[1] This computation was done using MAPLE — a programme for symbolic computation.

The example shows that it would be useful to have a criterion which allows to distinguish between f and g. The result which will be given below is such a criterion, which works in many cases. It is based on the relationship between the groups $\hat{U}$ and U.

Let f be a germ in $\mathcal{M}_{x\,\lambda}$ and $g(x, \lambda_1, \lambda_2) := f(x, \lambda_2, \lambda_1)$. We consider the tangent space $T(f, \hat{U})$. Its relationship with T(f, U) is given by

$$T(f, U) = T(f, \hat{U}) + \mathbb{R} \cdot \left\{ \lambda_2 f_{\lambda_1} \right\} .$$

Hence it is trivial to compute T(f, U) once $T(f, \hat{U})$ is known. It is obvious that the expression

$$T(f, \hat{U}) = \mathcal{E}_{x,\lambda} \left\{ x f, \lambda_1 f, \lambda_2 f, \lambda_1 f_x, \lambda_2 f_x, x^2 f_x \right\} + \\ \mathcal{E}_{\lambda} \left\{ \lambda_1^2 f_{\lambda_1}, \lambda_1 \lambda_2 f_{\lambda_1}, \lambda_2^2 f_{\lambda_1}, \lambda_1^2 f_{\lambda_2}, \lambda_1 \lambda_2 f_{\lambda_2}, \lambda_2^2 f_{\lambda_2} \right\}$$

is symmetric in the differential operators appearing with respect to exchanging λ_1 with λ_2 and $\partial/\partial\lambda_1$ and $\partial/\partial\lambda_2$. Let $w \in W$ be the equivalence interchanging λ_1 and λ_2 (See section II. 2.). Then $g(x, \lambda_1, \lambda_2) = f(x, \lambda_2, \lambda_1) = w \,.\, f$. From proposition II. 3. 9 we obtain

$$T(g, \hat{U}) = T(w \,.\, f, \hat{U}) = w \,.\, T(f, \hat{U}) . \tag{1.1}$$

This means that when $T(f, \hat{U})$ is already known, $T(g, \hat{U})$ is obtained by exchanging λ_1 with λ_2 in $T(f, \hat{U})$. Again it is then trivial to determine T(g, U) by adding the one-dimensional space $\mathbb{R} \cdot \{\lambda_2 g_{\lambda_1}\}$.

We now state the criterion.

1.2 Theorem. *Let f be a germ in $\mathcal{M}_{x,\lambda}$ of finite codimension. Then the following statements are equivalent:*

A) $P(f, \hat{U}) \subset P(f, U)$.

B) For all $p \in P(f, \hat{U})$

$$\lambda_2^k \frac{\partial^k p}{\partial \lambda_1^k} \in T(f, U)$$

for all $k \in \mathbb{N}_0$

1.3 Remark. Note that statement B) does not involve P(f, U). Hence it can be checked once $P(f, \hat{U})$ and T(f, U) are known. For this $\lambda_2{}^k \partial^k p/\partial\lambda_1{}^k \in T(f, U)$ has to be checked only for a finite number of integers k and for a finite number of germs p, since f is finitely-determined.

Theorem 1.2 can be used in the following way: Let f and g be defined as above. The first step is to calculate $T(f, \hat{U})$ and to determine $P(f, \hat{U}) = \mathrm{Itr}_{\hat{U}}\, T(f, \hat{U})$ by theorem III. 1. 4. This immediately yields $P(g, \hat{U})$ by exchanging λ_1 with λ_2 in $P(f, \hat{U})$, since $P(g, \hat{U}) = \mathrm{Itr}_{\hat{U}}\, T(g, \hat{U})$ and by equality (1.1). Applying the theorem it is possible to check, whether

$$P(f, \hat{U}) \subset P(f, U) \text{ or } P(g, \hat{U}) \subset P(g, U) \tag{1.2}$$

holds. The germ which the corresponding inclusion is satisfied for will then be chosen for calculating its U-higher-order terms. For many of the germs appearing in the classification theorem in section III. 6 one of the inclusions in (1.2) is satisfied.

1.4 Examples. 1. Consider $f = A x^3 + B x \lambda_1^2 + C \lambda_2$ and $g = A x^3 + B x \lambda_2^2 + C \lambda_1$.

where $A, B, C \neq 0$. Then

$$T(f, \bar{U}) = \mathcal{M}^4 + <\lambda_1, \lambda_2>^3 + \mathbb{R}\left\{ x^2\lambda_1, x^2\lambda_2, x\lambda_1\lambda_2, x\lambda_2^2, x\lambda_2, \lambda_1^2, \lambda_1\lambda_2, \lambda_2^2 \right\},$$

$$P(f, \hat{U}) = \mathcal{M}^4 + <\lambda_1, \lambda_2>^3 + \mathbb{R}\left\{ x^2\lambda_2, x\lambda_1\lambda_2, x\lambda_2^2, \lambda_1\lambda_2, \lambda_2^2 \right\},$$

In this case $T(f, U) = T(f, \bar{U})$ and it turns out that condition B) of theorem 1.2 is satisfied. To see this it is only necessary to check

$$\lambda_2^k \frac{\partial^k p}{\partial \lambda_1^k} \in T(f, U)$$

where p is one of the monomials

$$x^2\lambda_2, x\lambda_1\lambda_2, x\lambda_2^2, \lambda_1\lambda_2, \lambda_2^2 .$$

As a consequence $P(f, \hat{U}) \subset P(f, U)$. In fact, in this case $P(f, \hat{U}) = P(f, U)$.

For g we have

$$T(g, \bar{U}) = \mathcal{M}^4 + <\lambda_1, \lambda_2>^3 + \mathbb{R}\left\{ x^2\lambda_1, x^2\lambda_2, x\lambda_1\lambda_2, x\lambda_1^2, x\lambda_1, \lambda_1^2, \lambda_1\lambda_2, \lambda_2^2 \right\},$$

$$P(g, \hat{U}) = \mathcal{M}^4 + <\lambda_1, \lambda_2>^3 + \mathbb{R}\left\{ x^2\lambda_1, x\lambda_1\lambda_2, x\lambda_1^2, \lambda_1\lambda_2, \lambda_1^2 \right\}$$

and it turns out that condition B) is not satisfied. Hence $P(g, \hat{U}) \not\subset P(g, U)$ — in fact,

$$P(g, U) = \mathcal{M}^4 + <\lambda_1, \lambda_2>^3 .$$

2. Consider $f = x^3 + x\lambda_1^3 + \lambda_2$ and $g = x^3 + x\lambda_2^3 + \lambda_1$. For these germs neither $P(f, \hat{U}) \subset P(f, U)$ nor $P(g, \hat{U}) \subset P(g, U)$ holds. The codimension of f is two.

We now give the proof of theorem 1.2. First we state a lemma.

1.5 Lemma. *Let f be a germ in $\mathcal{M}_{x,\lambda}$ of finite codimension. Then the following statements are equivalent:*

A) $P(f, \hat{U}) \subset P(f, U)$.

B) There exists a U-intrinsic subspace V of $T(f, U)$ such that $P(f, \hat{U}) \subset V$.

C) For all $p \in P(f, \hat{U})$ $\quad U . p \subset T(f, U)$.

Proof: A) $\Rightarrow$ B): This is trivial, since P(f, U) is U-intrinsic.

B) $\Rightarrow$ C): Suppose $P(f, \hat{U}) \subset V \subset T(f, U)$ for a U-intrinsic vector space V. Then

$$U . V \subset V \subset T(f, U) \ ,$$

which implies C).

C) $\Rightarrow$ A): Condition C) implies

$$P(f, \hat{U})^U = \sum_{p \in P(f, \hat{U})} U . p \ \subset T(f, U) \ .$$

By proposition III. 1. 9 $P(f, \hat{U})^U$ is U-intrinsic. Hence it follows that

$$P(f, \hat{U})^U \subset \mathrm{Itr}_U T(f, U) = P(f, U) \ .$$

Hence

$$P(f, \hat{U}) \subset P(f, \hat{U})^U \subset P(f, U) \ . \square$$

Proof of theorem 1.2: Assume $P(f, \hat{U}) \subset P(f, U)$ holds and let $p \in P(f, \hat{U})$. It is sufficient to prove condition B) for the case, when p is a polynomial. To see this note that since f is finitely-determined T(f, U) can be written as

$$T(f, \hat{U}) = \mathcal{M}^k + V \ ,$$

where V is a finite-dimensional vector space. $\mathcal{M}^k$ is obviously $\hat{U}$-intrinsic, hence

$$\mathcal{M}^k \subset P(f, \hat{U}) \ .$$

Therefore we can write

$$p = \bar{p} + r \ ,$$

where $r \in \mathcal{M}^k$ and $\bar{p}$ is a polynomial. It is trivial to show that $\lambda_2 \dfrac{\partial^k r}{\partial \lambda_1^k}$ is in $\mathcal{M}^k$ and hence

$$\lambda_2^k \frac{\partial^k r}{\partial \lambda_2^k} \in \mathcal{M}^k \subset T(f, U)$$

for all $k \in \mathbb{N}_0$.

Now consider a polynomial $p \in P(f, \hat{U})$. By lemma 1.5 $U \cdot p \subset T(f, U)$. In particular $p(x, \lambda_1 + t\lambda_2, \lambda_2) \in T(f, U)$ for all $t \in \mathbb{R}$. Using the Taylor-expansion — which terminates — with respect to t at $t = 0$ we obtain

$$p(x, \lambda_1 + t\lambda_2, \lambda_2) = p(x, \lambda_1, \lambda_2) + \sum_{j=1}^{k} \frac{1}{j!} \lambda_2^j \frac{\partial^j}{\partial \lambda_1^j} p(x, \lambda_1, \lambda_2)\, t^j \qquad (1.3)$$

for some number $k \in \mathbb{N}_0$. Applying this formula for k pairwise distinct values $t_1, \ldots, t_k$ and using the abbreviations

$$w_i := p(x, \lambda_1 + t_i \lambda_2, \lambda_2) - p(x, \lambda_1, \lambda_2)$$

and

$$p_j = \frac{1}{j!} \lambda_2^j \frac{\partial^j}{\partial \lambda_1^j} p(x, \lambda_1, \lambda_2)$$

for $i, j = 1, \ldots, k$, we obtain the following linear system of equations:

$$\begin{pmatrix} t_1 & t_1^2 & \cdots & t_1^k \\ t_2 & t_2^2 & \cdots & t_2^k \\ \vdots & \vdots & & \vdots \\ t_k & t_k^2 & \cdots & t_k^k \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_k \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_k \end{pmatrix} .$$

Since $w_i \in T(f, U)$ for $i = 1, \ldots, k$ and the matrix in the system is invertible, it follows that

$$\lambda_2^j \frac{\partial^j p}{\partial \lambda_1^j}$$

is in $T(f, U)$ for $j = 1, \ldots, k$, which implies B).

To show the converse first note that it is again sufficient only to consider polynomials $p \in P(f, U)$ and an equivalence $u \in U$. According to proposition II. 2. 5 we can write $u = n\,\hat{u}$, where $n \in N$ and $\hat{u} \in \hat{U}$. We have

$$u \,.\, p = (n\,\hat{u}) \,.\, p$$

$$= n \,.\, (\hat{u} \,.\, p) \quad .$$

Since $P(f, \hat{U})$ is U-intrinsic, $\hat{u} \,.\, p$ is in $P(f, U)$. Define $\tilde{p} := \hat{u} \,.\, p$. Then

$$u \,.\, p(x, \lambda_1, \lambda_2) = n \,.\, \tilde{p}(x, \lambda_1, \lambda_2)$$

$$= \tilde{p}(x, \lambda_1 + t\,\lambda_2, \lambda_2)$$

for some $t \in \mathbb{R}$. By (1.2) it follows that $\tilde{p}(x, \lambda_1 + t\,\lambda_2, \lambda_2) \in T(f, U)$ and hence $u \,.\, p \in T(f, U)$. □

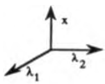
CHAPTER V

1. Geometrical description of two-parameter bifurcations

This section contains diagrams depicting the zero-sets and discriminants of the normal forms in the classification and their universal unfoldings. The discriminant associated to a given germ $f \in \mathcal{M}_{x,\lambda}$ is the following set

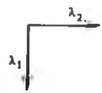
$$\left\{ (\lambda_1, \lambda_2) \in \mathbb{R}^2 \;\middle|\; \text{There exists } x \in \mathbb{R} \text{ such that } f(x, \lambda_1, \lambda_2) = \frac{\partial f}{\partial x}(x, \lambda_1, \lambda_2) = 0 \; . \right\}$$

The coordinates in the diagrams are oriented as follows:



Coordinates for the zero-sets

Fig. 1.1



Coordinates for the discriminants

Fig. 1.2

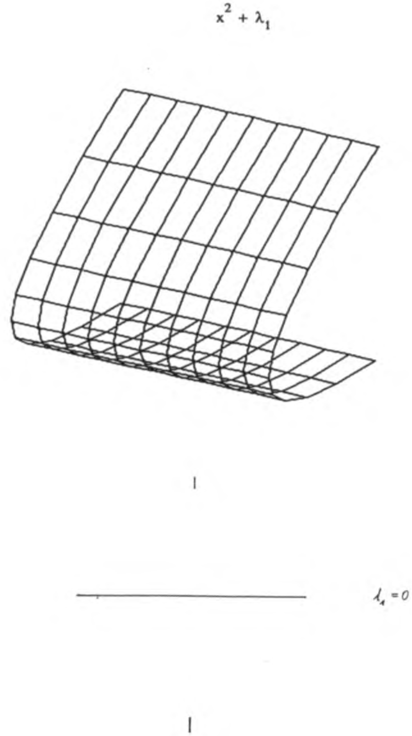
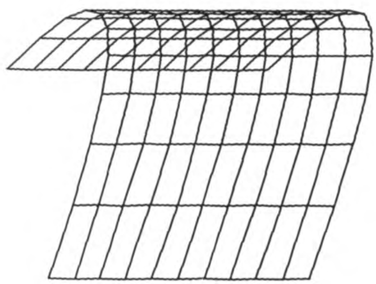
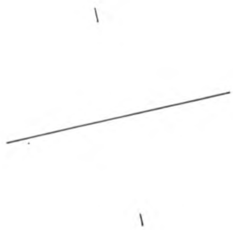


Fig. 1.3

$-x^2 + \lambda_1$

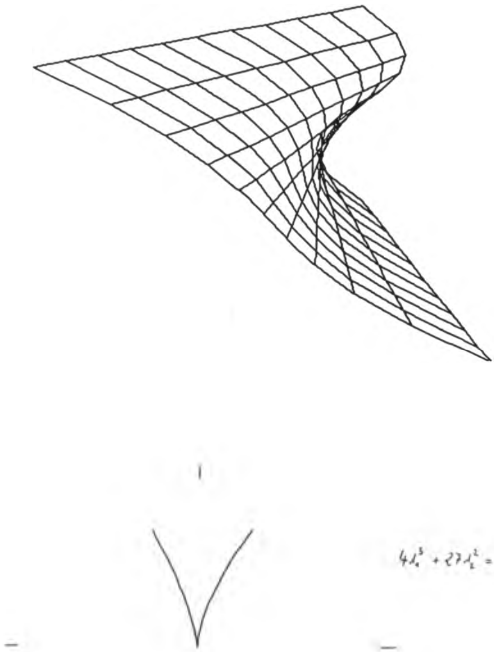




$\lambda_1 = 0$

Fig. 1.4

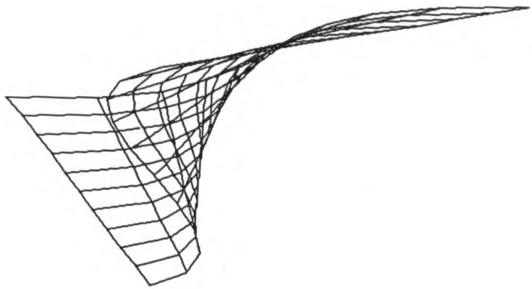
$$x^3 + x\lambda_1 + \lambda_2$$



$4\lambda_1^3 + 27\lambda_2^2 = 0$

Fig. 1.5

$$-x^3 + x\lambda_1 + \lambda_2$$



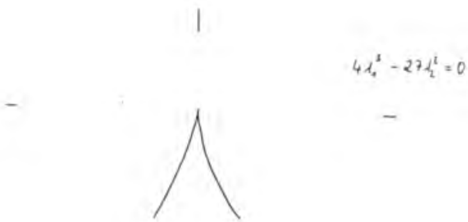


Fig. 1.6

$$x^2 + \lambda_1^2 + \lambda_2^2 + \alpha$$

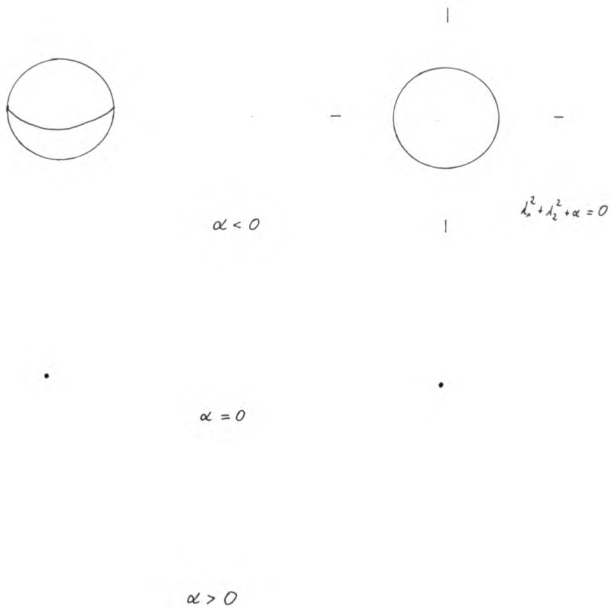


Fig. 1.7

$$x^2 - \lambda_1^2 - \lambda_2^2 + \alpha$$

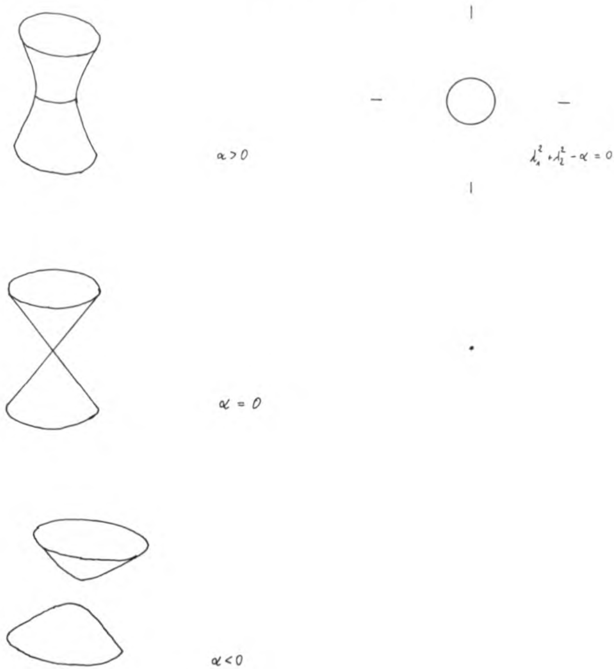


Fig. 1.8

$$-x^2 + \lambda_1^2 + \lambda_2^2 + \alpha$$

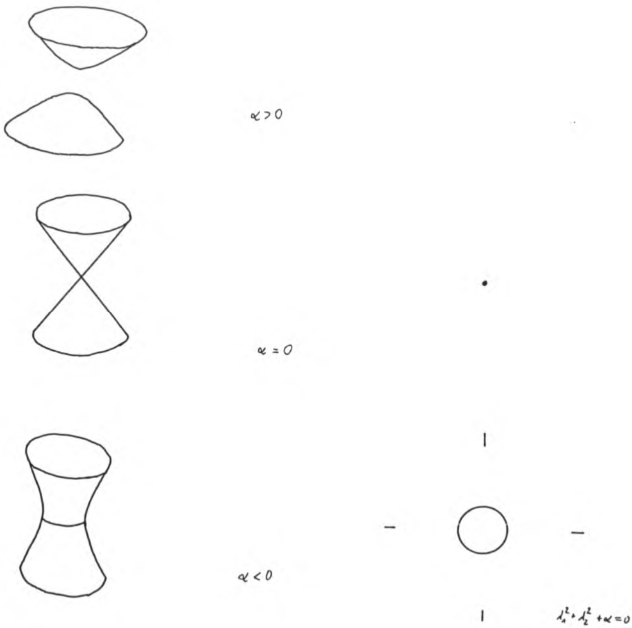


Fig. 1.9

$$-x^2 - \lambda_1^2 - \lambda_2^2 + \alpha$$

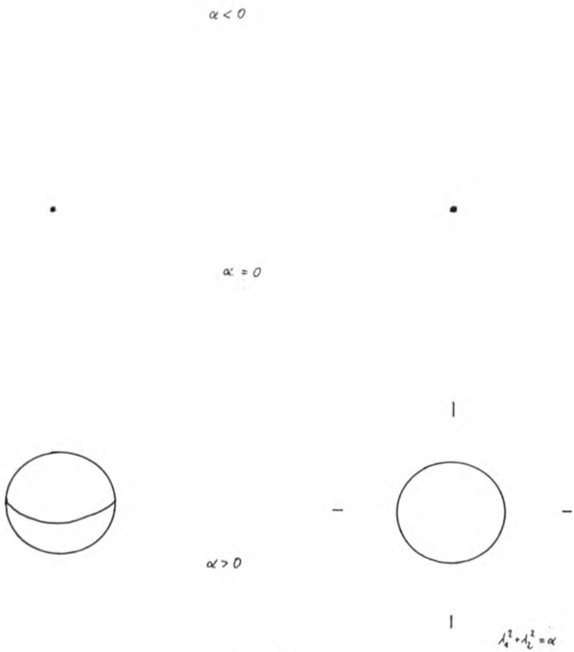


Fig. 1.10

$$x^3 + x\lambda_1^2 + \lambda_2 + \alpha x$$

$\alpha > 0$

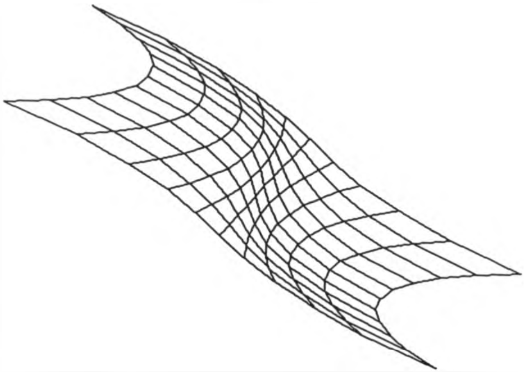


Fig. 1.11 a

$$x^3 + x\lambda_1^2 + \lambda_2 + \alpha x$$

$$\alpha = 0$$

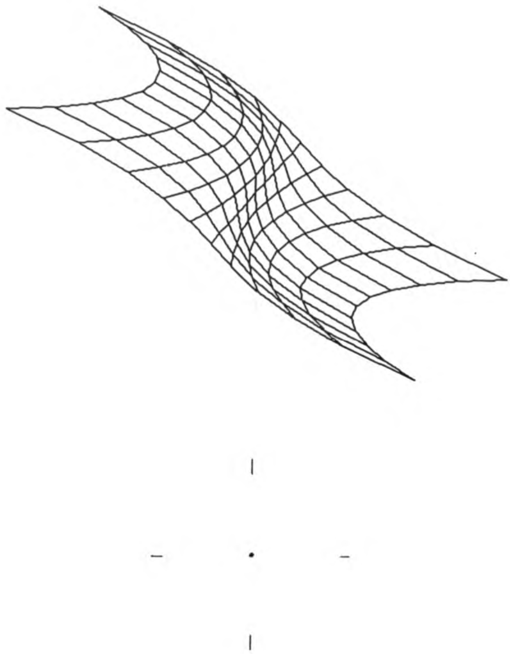
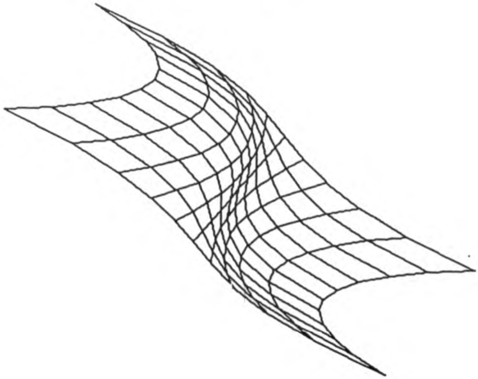


Fig. 1.11 b

$$x^3 + x\lambda_1^2 + \lambda_2 + \alpha x$$

$\alpha < 0$



$4(\lambda_1^2 + \alpha)^3 + 27\lambda_2^2 = 0$

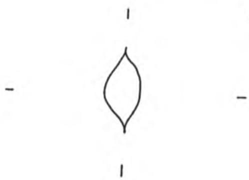
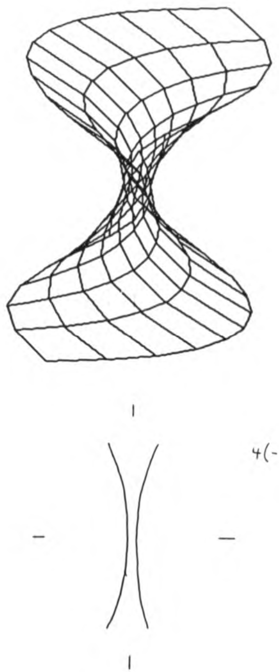


Fig. 1.11 c

$$x^3 - x\lambda_1^2 + \lambda_2 + \alpha x$$

$$\alpha < 0$$

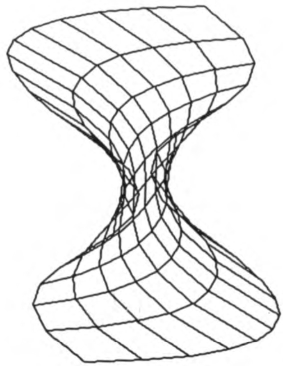


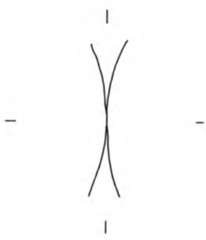
$4(-\lambda_1^2+\alpha)^3 + 27\lambda_2^2 = 0$

Fig. 1.12 a

$$x^3 - x\lambda_1^2 + \lambda_2 + \alpha x$$

$$\alpha = 0$$



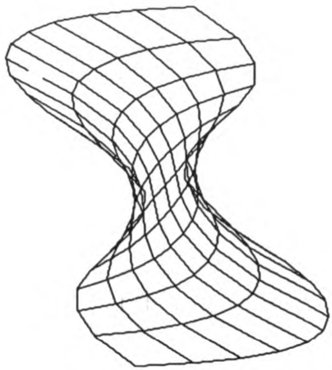


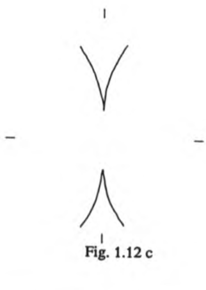
$-4\lambda_1^6 + 27\lambda_2^2 = 0$

Fig. 1.12 b

$$x^3 - x\lambda_1^2 + \lambda_2 + \alpha x$$

$$\alpha > 0$$



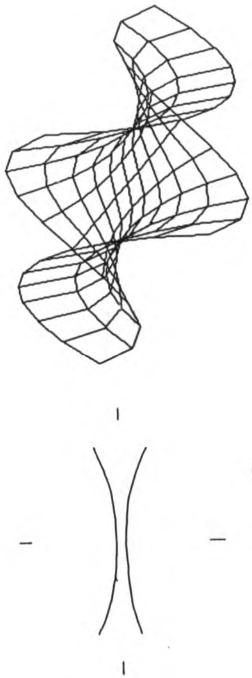


$-4(-\lambda_1^2+\alpha)^3+27\lambda_2^2=0$

Fig. 1.12 c

$$-x^3 + x\lambda_1^2 + \lambda_2 + \alpha x$$

$$\alpha < 0$$

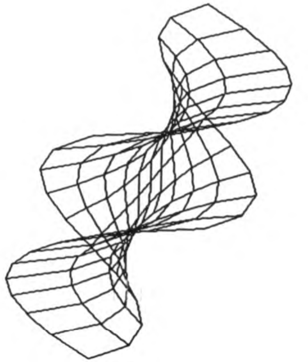


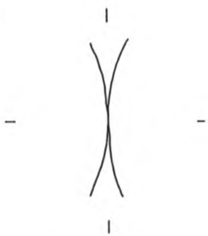
$$4(\lambda_1^2 + \alpha)^3 - 27\lambda_2^2 = 0$$

Fig. 1.13 a

$$-x^3 + x\lambda_1^2 + \lambda_2 + \alpha x$$

$$\alpha = 0$$



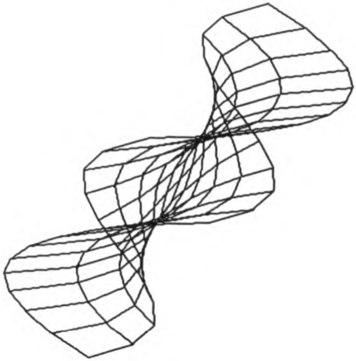


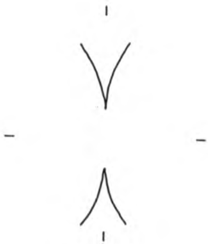
$4\lambda_1^6 - 27\lambda_2^2 = 0$

Fig. 1.13 b

$$-x^3 + x\lambda_1^2 + \lambda_2 + \alpha x$$

$$\alpha > 0$$





$4(\lambda_1^2+\alpha)^3 - 27\lambda_2^2 = 0$

Fig. 1.13 c

$$-x^3 - x\lambda_1^2 + \lambda_2 + \alpha x$$

$$\alpha < 0$$

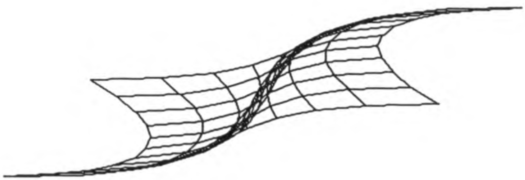


Fig. 1.14 a

$$-x^3 - x\lambda_1^2 + \lambda_2 + \alpha x$$

$$\alpha = 0$$

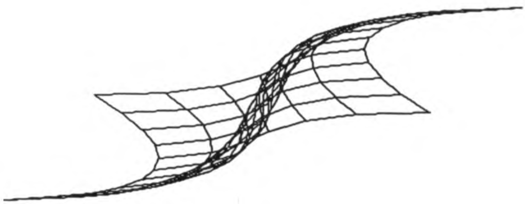
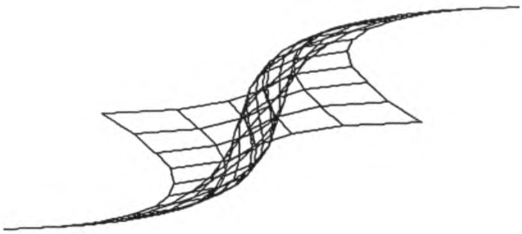


Fig. 1.14 b

$$-x^3 - x\lambda_1^2 + \lambda_2 + \alpha x$$

$\alpha > 0$



$4(-\lambda_1^2+\alpha)^3 - 27\lambda_2^2 = 0$

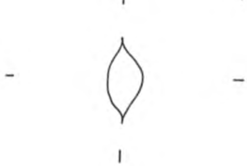
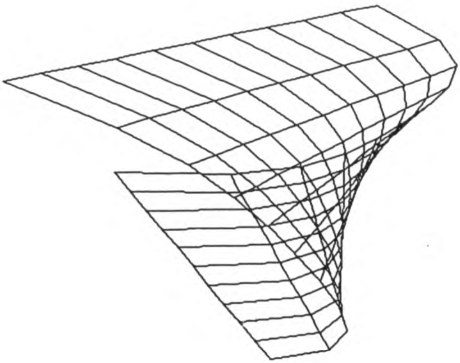


Fig. 1.14 c

$$x^4 + x\lambda_1 + \lambda_2 + \alpha x^2$$

$$\alpha < 0$$



$$4\alpha^3\lambda_1^2 + 27\lambda_1^4 - 16\alpha^4\lambda_2 + 128\alpha^2\lambda_2^2 - 144\alpha\lambda_1^2\lambda_2 - 256\lambda_2$$

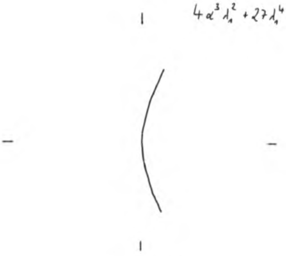
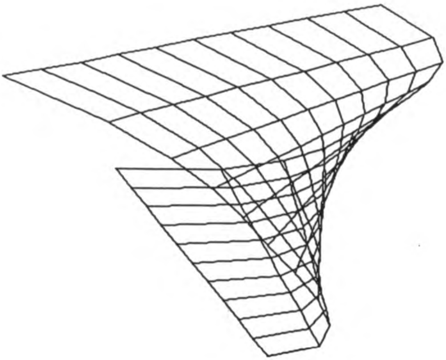
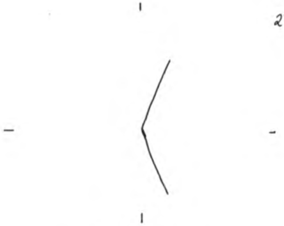


Fig. 1.15 a

$$x^4 + x\lambda_1 + \lambda_2 + \alpha x^2$$

$\alpha = 0$



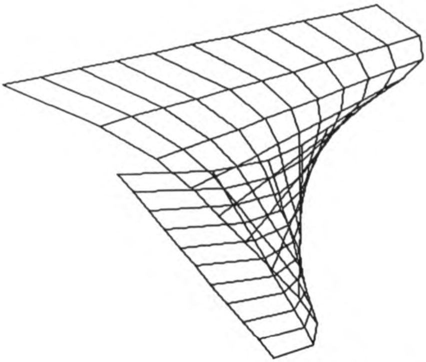


$27\lambda_1^4 - 256\lambda_2^3 = 0$

Fig. 1.15 b

$$x^4 + x\lambda_1 + \lambda_2 + \alpha x^2$$

$$\alpha > 0$$

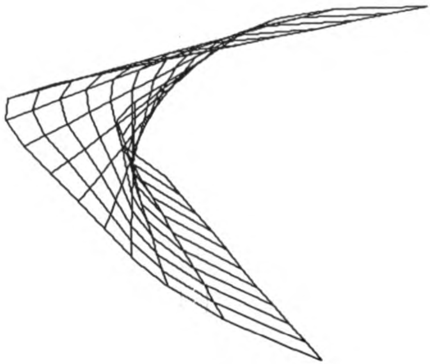


$$4\alpha^3\lambda_1^2 + 27\lambda_1^4 - 16\alpha^4\lambda_2 + 128\alpha^2\lambda_2^2 - 144\alpha\lambda_1^2\lambda_2 - 256\lambda_2^3 = 0$$

Fig. 1.15 c

$$-x^4 + x\lambda_1 + \lambda_2 + \alpha x^2$$

$\alpha < 0$



$$4\alpha^3\lambda_1^2 - 27\lambda_1^4 - 16\alpha^4\lambda_2 - 128\alpha^2\lambda_2^2 + 144\alpha\lambda_1^2\lambda_2 - 256\lambda_2^3 = 0$$

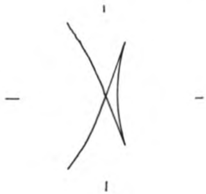
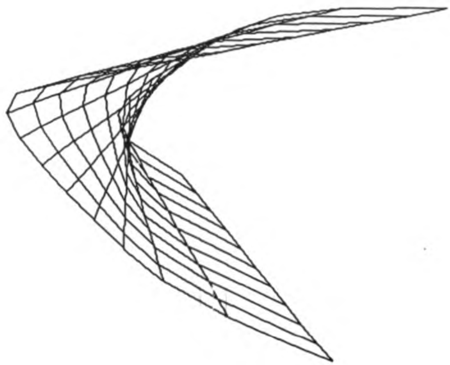


Fig. 1.16 a

$$-x^4 + x\lambda_1 + \lambda_2 + \alpha x^2$$

$$\alpha = 0$$



$27\lambda_1^4 + 256\lambda_2^3 = 0$

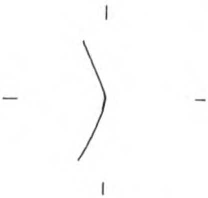
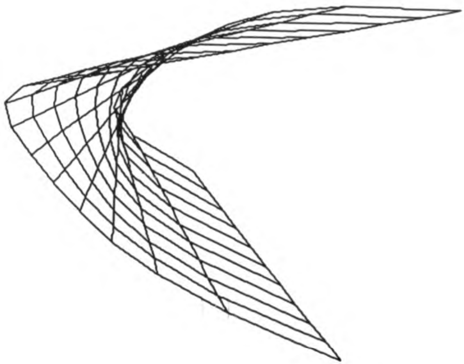


Fig. 1.16 b

$$-x^4 + x\lambda_1 + \lambda_2 + \alpha x^2$$

$$\alpha > 0$$



$$4\alpha^3\lambda_1^2 - 27\lambda_1^4 - 16\alpha^4\lambda_2 - 128\alpha^2\lambda_2^2 + 144\alpha\lambda_1^2\lambda_2 - 256\lambda_2^3 = 0$$

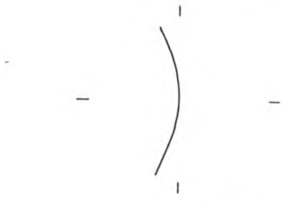


Fig. 1.16 c

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Part Two

Contents

Chapter 1

Classification of two-parameter bifurcations

1 Introduction

This chapter contains an extension of the work in part one of this thesis. The result is a classification of two-parameter bifurcations in one state variable up to codimension 3.

The classification theorem is stated in subsection 2.1. The normal forms are obtained by classifying the orbits arising from the action of the group of equivalences inductively on degree using Mather's lemma [Mat70]. The necessary determinacy results are then obtained by calculating the unipotent tangent spaces using results of Melbourne and Gaffney [Mel88], [Gaf86] based on work by Bruce, du Plessis and Wall [BdPW87]. A list of miniversal unfoldings of the germs in the classification theorem is also given.

Subsection 2.2 contains some examples of germs that have codimension greater than 3.

The work contained in part one of this thesis has meanwhile appeared in summarised form in [Pet91]. The normal forms given there also arise in a completely different context as part of another classification by Arnol'd in the paper [Arn76].

Notation in chapter 1 is the same as in part one.

2 The classification up to codimension 3

2.1 The classification theorem

Theorem 2.1.1 *Let $h \in \mathcal{E}_{x\lambda}$ be a germ satisfying $h = h_x = 0$. Let the codimension of h be less than or equal to 3. Then h is E-equivalent to one of the germs in table 1.1. Here $\varepsilon, \delta, \vartheta \in \{-1, +1\}$ and $\mu \in \mathbf{R} \setminus \{0\}$. The coefficient μ in the normal form $\mu x^7 + \varepsilon x^5 + x\lambda_2 + \lambda_1$ is a modal parameter.*

Proof. It will be shown how to derive the normal forms $\varepsilon x^3 + \delta x \lambda_2^4 + \lambda_1$ and $\varepsilon x^3 + x\lambda_1 + \delta\lambda_2^2$. The other cases can be treated similarly. Also note that the germs of the form $x^2 + \phi(\lambda_1, \lambda_2)$ have been classified by Izumiya up to codimension five in [Izu84].

Throughout the proof k-jets of germs $g \in \mathcal{E}_{x\lambda}$ are written as

$$j^k g = \sum_{\substack{p+q+r \le k \\ 0 \le p,q,r}} a_{pqr} x^p \lambda_1^q \lambda_2^r ,$$

where

$$a_{pqr} = \frac{1}{p!q!r!} \frac{\partial^{p+q+r} g}{\partial x^p \partial \lambda_1{}^q \partial \lambda_2{}^r}(0,0,0) .$$

Let $\mathcal{E}_{x\lambda}^k$ denote the space of k-jets of germs in $\mathcal{E}_{x\lambda}$.

A) Consider germs in $\mathcal{E}_{x\lambda}$ having non-vanishing 1-jet and satisfying $h = h_x = 0$. Without loss of generality $j^1 h = \lambda_1$ can be assumed. Now consider 2-jets of germs, whose 1-jets are equivalent to λ_1. By applying Mather's lemma [Mat70] one finds three orbits in $\mathcal{E}_{x\lambda}^2$ listed below together with their corresponding recognition conditions:

1. $a_{200} \neq 0 : \lambda_1 \pm x^2$

2. $a_{200} = 0, a_{101} \neq 0 : \lambda_1 + x\lambda_2$

3. $a_{200} = 0, a_{101} = 0 : \lambda_1$

germ	*codimension*
$\varepsilon x^2 + \lambda_1$	0
$\varepsilon x^3 + x\lambda_2 + \lambda_1$	0
$\varepsilon x^4 + x\lambda_2 + \lambda_1$	1
$\varepsilon x^2 + \lambda_1^2 - \lambda_2^2$	1
$\varepsilon x^2 + \delta(\lambda_1^2 + \lambda_2^2)$	1
$\varepsilon x^3 + \delta x\lambda_2^2 + \lambda_1$	1
$\varepsilon x^2 + \delta\lambda_1^2 + \lambda_2^3$	2
$\varepsilon x^3 + x\lambda_1 + \delta\lambda_2^2$	2
$\varepsilon x^3 + x\lambda_2^3 + \lambda_1$	2
$\varepsilon x^5 + \delta x^4 + x^2\lambda_2 + \lambda_1$	2
$\varepsilon x^2 + \delta\lambda_1^2 + \vartheta\lambda_2^4$	3
$\mu x^7 + \varepsilon x^5 + x\lambda_2 + \lambda_1$	3
$\varepsilon x^3 + \delta x\lambda_2^4 + \lambda_1$	3
$\varepsilon x^5 + \delta x^4 + \vartheta x\lambda_2^2 + \lambda_1$	3
$\varepsilon x^6 + \delta x^5 + x^2\lambda_2 + \lambda_1$	3
$\varepsilon x^7 + \delta x^4 + x^2\lambda_2 + \lambda_1$	3

Table 1.1: Normal forms for the germs up to codimension 3

Consider the last case. The orbits in $\mathcal{E}^3_{x\lambda}$ for germs whose 2-jets are equivalent to λ_1 are given by

1. $a_{300} \neq 0, D \neq 0:\ \lambda_1 \pm x^3 \pm x\lambda_2^2$
2. $a_{300} = 0, D \neq 0:\ \lambda_1 + x^2\lambda_2$
3. $a_{300} \neq 0, D = 0:\ \lambda_1 \pm x^3$
4. $a_{300} = 0, D = 0:\ \lambda_1 \pm x\lambda_2^2$

Here

$$D := 6a_{300}a_{102} - 2a_{201}^2 .$$

Consider the third case. The orbits in $\mathcal{E}^4_{x\lambda}$ for germs whose 3-jets are equivalent to $\lambda_1 \pm x^3$ are

1. $a_{103} \neq 0:\ \lambda_1 \pm x^3 + x\lambda_2^3$
2. $a_{103} = 0:\ \lambda_1 \pm x^3$

Taking the second case one finds for the orbits in $\mathcal{E}^5_{x\lambda}$ of germs whose 4-jets are equivalent to $\lambda_1 \pm x^3$:

1. $a_{104} \neq 0:\ \lambda_1 \pm x^3 \pm x\lambda_2^4$
2. $a_{104} = 0:\ \lambda_1 \pm x^3$

The second case leads to germs of codimension greater than 3. In the first case the germ $\lambda_1 \pm x^3 \pm x\lambda_2^4$ is 5-determined and has codimension 3.

B) Consider germs in $\mathcal{E}_{x\lambda}$ having vanishing 1-jet. Mather's lemma yields the following orbits in $\mathcal{E}^2_{x\lambda}$:

1. $a_{200} \neq 0, K \neq 0:\ \pm x^2 + \lambda_1^2 - \lambda_2^2,\ \pm x^2 \pm (\lambda_1^2 + \lambda_2^2)$
2. $a_{200} \neq 0$, A or $D \neq 0:\ \pm x^2 \pm \lambda_1^2$
3. $a_{200} \neq 0, A = 0, D = 0:\ \pm x^2$
4. $a_{200} = 0, L \neq 0:\ x\lambda_1 \pm \lambda_2^2$

5. $a_{200} = 0$, $L = 0$, a_{110} or $a_{101} \neq 0$: $x\lambda_1$

6. $a_{200} = 0, a_{110} = 0, a_{101} = 0, D^* \neq 0$: $\pm(\lambda_1^2 + \lambda_2^2)$, $\lambda_1^2 - \lambda_2^2$

7. $a_{200} = 0, a_{110} = 0, a_{101} = 0, D^* = 0$, a_{020} or $a_{002} \neq 0$: $\pm\lambda_1^2$

8. $a_{200} = 0, a_{110} = 0, a_{101} = 0, D^* = 0, a_{020} = 0, a_{002} = 0$: 0

Here

$$
\begin{aligned}
A &:= -4a_{020}a_{200} + a_{110}^2 ,\\
D &:= -4a_{002}a_{200} + a_{101}^2 ,\\
K &:= a_{011}^2a_{200} - a_{011}a_{110}a_{101} - 4a_{200}a_{020}a_{002} + a_{020}a_{101}^2 + a_{110}^2a_{002} ,\\
L &:= a_{002}a_{110}^2 - a_{101}a_{011}a_{110} + a_{101}^2a_{020} ,\\
D^* &:= a_{011}^2 - 4a_{002}a_{020} .
\end{aligned}
$$

Consider the fourth case. The orbits in $\mathcal{E}^3_{x,\lambda}$ of germs whose 2-jets are equivalent to $x\lambda_1 \pm \lambda_2^2$ are

1. $a_{300} \neq 0$: $\pm x^3 + x\lambda_1 \pm \lambda_2^2$

2. $a_{300} = 0, a_{201} \neq 0$: $x\lambda_1 \pm \lambda_2^2 + x^2\lambda_2$

3. $a_{300} = 0, a_{201} = 0$: $x\lambda_1 \pm \lambda_2^2$

Proceeding further in the second and third case leads to germs of codimension greater than 3. The germ $\pm x^3 + x\lambda_1 \pm \lambda_2^2$ is 3-determined and has codimension 2. □

Corollary 2.1.2 *Miniversal unfoldings of the germs in theorem 2.1.1 can be chosen as listed in table 1.2.*

Proof. It will be shown how to derive miniversal unfoldings for the germs

$$\varepsilon x^7 + \delta x^4 + x^2\lambda_2 + \lambda_1$$

and

$$\varepsilon x^5 + \delta x^4 + x^2\lambda_2 + \lambda_1 .$$

germ	*unfolding terms*
$\varepsilon x^2 + \lambda_1$	—
$\varepsilon x^3 + x\lambda_2 + \lambda_1$	—
$\varepsilon x^4 + x\lambda_2 + \lambda_1$	x^2
$\varepsilon x^2 + \lambda_1^2 - \lambda_2^2$	1
$\varepsilon x^2 + \delta(\lambda_1^2 + \lambda_2^2)$	1
$\varepsilon x^3 + \delta x\lambda_2^2 + \lambda_1$	x
$\varepsilon x^2 + \delta\lambda_1^2 + \lambda_2^3$	$1, \lambda_2$
$\varepsilon x^3 + x\lambda_1 + \delta\lambda_2^2$	$1, \lambda_1$
$\varepsilon x^3 + x\lambda_2^3 + \lambda_1$	$x, x\lambda_2$
$\varepsilon x^5 + \delta x^4 + x^2\lambda_2 + \lambda_1$	$x, x\lambda_2$
$\varepsilon x^2 + \delta\lambda_1^2 + \vartheta\lambda_2^4$	$1, \lambda_2, \lambda_2^2$
$\mu x^7 + \varepsilon x^5 + x\lambda_2 + \lambda_1$	x^2, x^3, x^7
$\varepsilon x^3 + \delta x\lambda_2^4 + \lambda_1$	$x, x\lambda_2, x\lambda_2^2$
$\varepsilon x^5 + \delta x^4 + \vartheta x\lambda_2^2 + \lambda_1$	$x, x^2, x^2\lambda_2$
$\varepsilon x^6 + \delta x^5 + x^2\lambda_2 + \lambda_1$	$x, x\lambda_1, x\lambda_2$
$\varepsilon x^7 + \delta x^4 + x^2\lambda_2 + \lambda_1$	$x, x\lambda_2^2, x^3$

Table 1.2: Miniversal unfoldings of the normal forms

A) Let $g = \varepsilon x^7 + \delta x^4 + x^2\lambda_2 + \lambda_1$. The germ g is 7-determined. This implies

$$\mathcal{M}^8 \subset T_e(g),$$

where

$$T_e(g) = \mathcal{E}_{x\lambda}\left\{\varepsilon x^7 + \delta x^4 + x^2\lambda_2 + \lambda_1, 7\varepsilon x^6 + 4\delta x^3 + 2x\lambda_2\right\} + \mathcal{E}_\lambda\left\{1, x^2\right\} .$$

A calculation shows that

$$\begin{aligned} T_e(g) \quad &= \quad \mathcal{M}^5 + < \lambda_1, \lambda_2 >^4 + \mathbf{R}\left\{x^\alpha \lambda_1^\beta \lambda_2^\gamma : \alpha \in \{0, 2\}\right\} \\ &\quad + \mathbf{R}\left\{x^3\lambda_1, x^3\lambda_2, x\lambda_1^2, x\lambda_1\lambda_2, x\lambda_2^2, x\lambda_1, 2x\lambda_2 + 4\delta x^3\right\} . \end{aligned}$$

Hence the only monomials $x^\alpha\lambda_1^\beta\lambda_2^\gamma$ not contained in $T_e(g)$ are $x^3, x\lambda_2^3, x\lambda_2$ and x . Since $2x\lambda_2 + 4\delta x^3 \in T_e(g)$, it follows that $\operatorname{codim}(g) = 3$ and that $x, x\lambda_2^3$ and x^3 can be chosen to yield a miniversal unfolding of g .

B) Let $g = \lambda_1 + x^2\lambda_2 + \delta x^4 + \varepsilon x^5$. The germ g is 5-determined, which can be shown by using the preparation theorem. Since this method is described in a much more complicated case in chapter 2 in the proof of lemma 3.5.3 the details are omitted here.

A calculation shows that

$$\begin{aligned} T_e(g) \quad &= \quad \mathcal{M}^4 + \mathcal{M}^2 < \lambda_1, \lambda_2 > + < \lambda_1, \lambda_2 >^2 \\ &\quad + \mathbf{R}\left\{x^2, x\lambda_1, 2x\lambda_2 + 4\delta x^3, \lambda_1, \lambda_2, 1\right\} . \end{aligned}$$

It follows that g has codimension 2 and that x and $x\lambda_2$ can be chosen as unfolding terms. □

Remark 2.1.3 It is possible to obtain the same result as in the preceding proof for $\varepsilon x^5 + \delta x^4 + x^2\lambda_2 + \lambda_1$ in a different way using a theorem of Mond and Montaldi [MM91]. One can check by explicit coordinate changes that g is equivalent to

$$\bar{g} = \lambda_1 + \frac{1}{4}\varepsilon x\lambda_2^2 + x^2\lambda_2 + x^4 .$$

One can think of $\bar{g}$ as being induced by the mapping

$$\gamma : \quad (\lambda_1, \lambda_2) \longrightarrow \left(\lambda_2, \frac{1}{4}\varepsilon\lambda_2^2, \lambda_1\right)$$

into the space (a, b, c) of unfolding parameters of a $\mathcal{K}$-miniversal unfolding of x^4 given by $x^4 + ax^2 + bx + c$. The theorem of Mond and Montaldi states that $T\mathcal{K}_V.\gamma$ and $T_e(g)$ have isomorphic normal spaces. Here $T\mathcal{K}_V.\gamma$ denotes the tangent space of the mapping γ with respect to $\mathcal{K}_V$-equivalence, which preserves the discriminant of $x^4 + ax^2 + bx + c$ — the swallowtail. Using the explicit formula for $T\mathcal{K}_V.\gamma$ leads to the same result as given above for the codimension and unfolding of g.

2.2 Some additional information

This subsection contains some examples concerning germs of codimension greater than 3.

Example 2.2.1 Consider the family of germs $g_k := \varepsilon x^{k+1} + x\lambda_2 + \lambda_1$, where $k \geq 2$ and $\varepsilon \in \{-1, +1\}$. All germs g_k are finitely-determined and

$$\operatorname{codim}(g_k) = \frac{(k-2)(k-1)}{2}, \tag{2.1}$$

g_k is $k+1$-determined only for $k = 2$ and $k = 3$.

The first step to show that 2.1 holds is to obtain the formula

$$\operatorname{codim}(g_k) = \dim_{\mathbf{R}} \frac{\mathcal{E}_x}{\mathcal{E}_{x^{k+1},x^k}\{1, x\}} \tag{2.2}$$

by generalising the reasoning used in example 3.6.2 of part one of this thesis. To determine $\mathcal{E}_{x^{k+1},x^k}\{1, x\}$ define two sets X_0 and X_1 by

$$\begin{aligned} X_0 &:= \{pk + q(k+1) : p, q \in \mathbf{N}_0\}, \\ X_1 &:= \{pk + q(k+1) + 1 : p, q \in \mathbf{N}_0\} \end{aligned}$$

and let $X := X_0 \cup X_1$. Also let $I_m := \{n \in \mathbf{N}_0 : n \geq m\}$. The set X has the following property: Suppose X contains a subset of k successive integers, $\{m, m+1, \ldots, m+(k-1)\}$ say, for some $m \in \mathbf{N}_0$. Then $I_m \subset X$. It follows that it is sufficient for $I_m \subset X$ to hold that X_0 contains the subset $\{m, m+1, \ldots, m+(k-2)\}$ of $k-1$ successive integers.

The set X_0 can be written as

$$X_0 = \bigcup_{s \geq 0} A_s \,, \tag{2.3}$$

where

$$A_s = \{pk + q(k+1) : p + q = s;\ p, q \in \mathbf{N}_0\} \ .$$

Note that the sets A_s are pairwise disjoint and that A_s consists of $s+1$ successive integers, since

$$A_s = \{sk, sk+1, \ldots, sk+s\} \ .$$

Hence $k-2$ is the smallest value of s such that A_s consists of $k-1$ successive integers. This implies

$$I_{(k-2)k} \subset X \,,$$

showing that

$$\mathcal{M}_x^{(k-2)k} \subset \mathcal{E}_{x^{k+1}, x^k}\{1, x\} \,.$$

It follows by 2.2 that

$$\operatorname{codim}(g_k) \leq (k-2)k < \infty \ .$$

To determine the precise value of $\operatorname{codim}(g_k)$ note that by the description of X_0 given in 2.3, $\mathbf{N}_0 \setminus X$ is the disjoint union of the sets

$$G_s := \{(s-1)(k+1) + 2, \ldots, sk - 1\}$$

for $s = 1, \ldots, k-2$. Therefore

$$\operatorname{codim}(q_k) = \sum_{s=1}^{k-2} \# G_s \,,$$

and since $\# G_s = k - s - 1$, the result is

$$\operatorname{codim}(g_k) = \frac{(k-2)(k-1)}{2} \,.$$

It follows from

$$\mathbf{N}_0 \setminus X = \bigcup_{s=1}^{k-2} G_s$$

that the germs $g_k = \varepsilon x^{k+1} + x\lambda_2 + \lambda_1$ are $k+1$-determined only for $k = 2$ and $k = 3$.

Example 2.2.2 Let $k \geq 1$ and consider the germs $h_k := \varepsilon x^3 + \delta x \lambda_2^k + \lambda_1$ for even k and $h_k := \varepsilon x^3 + x\lambda_2^k + \lambda_1$ for odd k, where $\varepsilon \in \{-1, +1\}$. Then $\operatorname{codim}(h_k) = k - 1$ and the terms $x, x\lambda_2, \ldots, x\lambda_2^{k-2}$ yield a miniversal unfolding of h_k ,.

Example 2.2.3 Let $k \geq 2$ and consider the germs $p_k := \varepsilon x^3 + x\lambda_1 + \delta\lambda_2^k$ for even k and $p_k := \varepsilon x^3 + x\lambda_1 + \lambda_2^k$ for odd k, where $\varepsilon \in \{-1, +1\}$. Then $\operatorname{codim}(p_k) = k$ and the terms $1, \lambda_1$ yield a miniversal unfolding of p_k in the case $k = 2$ and the terms $1, \lambda_1, \lambda_2, \ldots, \lambda_2^{k-2}$ for $k > 2$.

Example 2.2.4 The germs $\varepsilon x^4 + \delta x\lambda_2^2 + \lambda_1$ and $\varepsilon x^3 + \delta x\lambda_1^2 + \lambda_1^2 + \lambda_2^2$, where $\varepsilon, \delta \in \{-1, +1\}$, both have codimension 4. Miniversal unfoldings are given by the terms $x^2, x^2\lambda_2^2, x, x\lambda_1$ inthe first and by $x, x\lambda_2, 1, \lambda_1$ in the second case.

Chapter 2

Equivariant bifurcations with group action on state and parameter space

1 Introduction

This chapter is devoted to to a generalisation of the singularity theory approach to equivariant bifurcation theory. In this context one studies the equivalence relation of parametrised contact equivalence on a set of mappings $\mathbf{R}^n \times \mathbf{R}^k \longrightarrow \mathbf{R}^n$, where $\mathbf{R}^n$ is referred to as the state space and $\mathbf{R}^k$ as the parameter space. These concepts were introduced by Golubitsky and Schaeffer [GS79b, GS79a]. Many cases have been studied, in which the bifurcations are equivariant, see e. g. [Mel86], [Mel88], [Mel87], [GR87], [GS84], [GSS88], [Ste88]. The property of equivariance is defined via a group action of a compact Lie group Γ on the state space. The aim of this chapter is to study cases where the group Γ acts on the parameter space as well. This type of group action has been studied in a somewhat different context by Janeczko and Roberts [JR91], who use the theory of Lagrangian singularities to classify symmetric caustics. (See also [JR] for their work on this topic.)

The problems treated in this chapter are certain $\mathbf{Z}_2$-equivariant and $\mathbf{D}_4$-

equivariant bifurcations. The first is a simple example treated to show that the general theory outlined in section 2 works. The case of $\mathbf{D}_4$-equivariant bifurcations forms the main example and is treated in section 3. The group action of $\mathbf{D}_4$ defined there is motivated by a problem in physics having this particular symmetry.

2 General definitions and background

2.1 Notation

The following is a list of notation used in chapter 2. More notation will be defined within the text.

Coordinates in the state space $\mathbf{R}^n$ are denoted by $x := (x_1, \ldots, x_n)$ and coordinates in the parameter space $\mathbf{R}^k$ by $\lambda := (\lambda_1, \ldots, \lambda_k)$.

$\mathcal{E}_{u_1,\ldots,u_m}$ denotes the ring of real-valued C^∞-function germs in the variables $u_1, \ldots, u_m$ at $(0, \ldots, 0)$. $\mathcal{M}_{u_1,\ldots,u_m}$ denotes the maximal ideal in $\mathcal{E}_{u_1,\ldots,u_m}$.

Let Γ be a compact Lie group acting linearly on $\mathbf{R}^n$ and let $Hom_\Gamma(\mathbf{R}^n)$ denote the set of linear maps $\mathbf{R}^n \longrightarrow \mathbf{R}^n$ that commute with the action of Γ. Then $\mathcal{L}(\Gamma)^o$ is defined to be the connected component containing the identity map of $Hom_\Gamma(\mathbf{R}^n) \cap GL(n, \mathbf{R})$.

The identity matrix in $GL(n, \mathbf{R})$ is denoted by I_n. The trivial group is denoted by $\mathbf{1}$.

The symbol $\sim$ is used to denote Γ-equivalence. See definition 2.2.4.

2.2 Definitions and determinacy theorems

Definition 2.2.1 Let Γ be a compact Lie group acting on $\mathbf{R}^n \times \mathbf{R}^k$ by

$$\gamma.(x, \lambda) = (\gamma x, \gamma \lambda),$$

i. e. Γ has one representation on the state space $\mathbf{R}^n$ and another one on the parameter space $\mathbf{R}^k$.

A) A smooth map germ $g : \mathbf{R}^n \times \mathbf{R}^k \longrightarrow \mathbf{R}^n$ at $(0,0)$ is said to be Γ-*equivariant*, if

$$g(\gamma x, \gamma\lambda) = \gamma \cdot g(x, \lambda)$$

for all $\gamma \in \Gamma$, $x \in \mathbf{R}^n$ and $\lambda \in \mathbf{R}^k$.

B) A smooth function germ $f : \mathbf{R}^n \times \mathbf{R}^k \longrightarrow \mathbf{R}$ at $(0,0)$ is said to be Γ-*invariant*, if

$$f(\gamma x, \gamma\lambda) = f(x, \lambda)$$

for all $\gamma \in \Gamma$, $x \in \mathbf{R}^n$ and $\lambda \in \mathbf{R}^k$.

Remark 2.2.2 Let a particular group action of Γ be fixed. Then the set of Γ-invariant function germs forms a ring denoted by $\mathcal{E}_{x\lambda}(\Gamma)$. The set of Γ-equivariant map germs has the structure of an $\mathcal{E}_{x\lambda}(\Gamma)$-module and is denoted by $\vec{\mathcal{E}}_{x\lambda}(\Gamma)$.

Definition 2.2.3 Let $g \in \vec{\mathcal{E}}_{x\lambda}(\Gamma)$. If g satisfies

$$g(0,0) = 0 \quad \text{and} \quad (D_x g)(0,0) = 0,$$

it is called a *bifurcation problem*.

Definition 2.2.4 Two bifurcation problems $g, h \in \vec{\mathcal{E}}_{x\lambda}(\Gamma)$ are said to be Γ-*equivalent*, if there exist Γ-equivariant diffeomorphism germs R and S satisfying the conditions given below such that

$$h = S \cdot g \circ R.$$

R is a diffeomorphism germ $\mathbf{R}^n \times \mathbf{R}^k \longrightarrow \mathbf{R}^n \times \mathbf{R}^k$ at $(0,0)$ of the form

$$R(x, \lambda) = (X(x, \lambda), \Lambda(\lambda)),$$

where X is a smooth map germ $\mathbf{R}^n \times \mathbf{R}^k \longrightarrow \mathbf{R}^n$ at $(0,0)$ satisfying

$$X(0,0) = 0$$

and

$$D_x X(0,0) \in \mathcal{L}(\Gamma)^\circ;$$

and where Λ is a smooth map germ $\mathbf{R}^l \longrightarrow \mathbf{R}^k$ at 0 satisfying

$$\Lambda(0) = 0$$

and

$$\det(D\Lambda(0)) \neq 0 .$$

S is a diffeomorphism germ $\mathbf{R}^n \times \mathbf{R}^k \longrightarrow GL(n, \mathbf{R})$ at $(0,0)$ satisfying

$$S(0,0) \in \mathcal{L}(\Gamma)^{\circ} .$$

Remark 2.2.5 The property of Γ-equivariance for S can be restated as

$$S(\gamma x, \gamma\lambda) = \gamma . S(x, \lambda)\gamma^{-1} ,$$

i. e. the action of Γ on $GL(n, \mathbf{R})$ is defined by

$$\gamma . M = \gamma . M\gamma^{-1}$$

for all $\gamma \in \Gamma$ and $M \in GL(n, \mathbf{R})$. The set of all Γ-equivariant matrix-valued germs $\mathbf{R}^n \times \mathbf{R}^k \longrightarrow GL(n, \mathbf{R})$ is denoted by $\overleftrightarrow{\mathcal{E}}_{x\lambda}(\Gamma)$. The condition of Γ-equivariance for R can be restated as

$$X(\gamma x, \gamma\lambda) = \gamma X(x, \lambda)$$

$$\Lambda(\gamma\lambda) = \gamma\Lambda(\lambda) .$$

Using these statements it is easy to show that the set of all Γ-equivalences forms a group denoted by E, in which multiplication is defined in the standard way. (Compare [GSS88, Mel88] or part 1 of this thesis.)

Definition 2.2.6 Let $g \in \overrightarrow{\mathcal{E}}_{x\lambda}(\Gamma)$. Then

$$T_e^{\Gamma}(g) := \overleftrightarrow{\mathcal{E}}_{x\lambda}(\Gamma)g + (D_x g)\overrightarrow{\mathcal{E}}_{x\lambda}(\Gamma) + (D_\lambda g)\overrightarrow{\mathcal{E}}_{\lambda}(\Gamma)$$

is called the *extended tangent space* of the germ g. The number

$$\operatorname{codim}^{\Gamma}(g) := \dim_{\mathbf{R}} \frac{\overrightarrow{\mathcal{E}}_{x\lambda}(\Gamma)}{T_e^{\Gamma}(g)}$$

is said to be the Γ-*codimension* of the germ g.

The following is the determinacy result for Γ-equivalence due to Damon [Dam84].

Theorem 2.2.7 *Let $g \in \vec{\mathcal{E}}_{x\lambda}(\Gamma)$. Then g is finitely-determined with respect to Γ-equivalence if and only if $T_e^{\Gamma}(g)$ has finite codimension in $\vec{\mathcal{E}}_{x\lambda}(\Gamma)$.*

Proof. This follows by slightly modifying Damon's result to account for the action of Γ on the parameter space $\mathbf{R}^k$. (Compare theorem 10.2 in [Dam84].) □

Definition 2.2.8 Let $(S, R) = (S, X, \Lambda)$ be a Γ-equivalence as defined in definition 2.2.4. The subgroup U consisting of all Γ-equivalences satisfying the additional conditions

$$\begin{aligned} S(0,0) &= I_n \\ D_x X(0,0) &= I_n \\ D_\lambda \Lambda(0) &= I_k \end{aligned}$$

is called the *subgroup of unipotent Γ-equivalences.*

Definition 2.2.9 Let $g \in \vec{\mathcal{E}}_{x\lambda}(\Gamma)$. Then

$$RT^{\Gamma}(g, U) := \mathcal{M}\vec{\mathcal{E}}_{x\lambda}(\Gamma)g + (D_x g)(\mathcal{M}_x^2 + \mathcal{M}_\lambda)\vec{\mathcal{E}}_{x\lambda}(\Gamma)$$

is called the *restricted unipotent Γ-tangent space* of the germ g and

$$T^{\Gamma}(g, U) := RT^{\Gamma}(g, U) + (D_\lambda g)\mathcal{M}_\lambda^2 \vec{\mathcal{E}}_\lambda(\Gamma)$$

is called the *unipotent Γ-tangent space* of the germ g.

Remark 2.2.10 The unipotent Γ-tangent space $T^{\Gamma}(g, U)$ corresponds to the group U of unipotent Γ-equivalences. It has finite codimension if and only if $T_e^{\Gamma}(g)$ has.

Definition 2.2.11 Let $g \in \vec{\mathcal{E}}_{x\lambda}(\Gamma)$. The following $\mathcal{E}_{x\lambda}(\Gamma)$-submodule of $\vec{\mathcal{E}}_{x\lambda}(\Gamma)$

$$P(g) = \left\{ p \in \vec{\mathcal{E}}_{x\lambda}(\Gamma) : h + p \sim g \text{ for all } h \in \vec{\mathcal{E}}_{x\lambda}(\Gamma) \text{ satisfying } h \sim g \right\}$$

is called the *module of higher-order terms* of the germ g.

Definition 2.2.12 Let E be the group of Γ-equivalences and let V be a vector subspace of $\overline{\mathcal{E}}_{x\lambda}(\Gamma)$.
A) V is said to be *intrinsic*, if it is invariant under the action of E, i. e. if $E.V = V$.
B) The vector space

$$\mathbf{Itr}V := \bigcap_{e \in E} e.V$$

is called the *intrinsic part* of V.

The following theorem due to Gaffney [Gaf86] based on work by Bruce, du Plessis and Wall [BdPW87] is a determinacy result for Γ-equivalence.

Theorem 2.2.13 *Let $g \in \overline{\mathcal{E}}_{x\lambda}(\Gamma)$ be a germ of finite codimension. Then*

$$P(g) \supset \mathbf{Itr}\left(T^{\Gamma}(g, U)\right).$$

Proof. The proof of this result is analogous to the one given in [Gaf86] in the case of ordinary Γ-equivalence. □

2.3 An example with $\mathbf{Z}_2$-symmetry

In this subsection a simple example for the general theory outlined above is considered. Let a group action of $\Gamma = \mathbf{Z}_2$ be defined by

$$-1.(x, \lambda) := (-x, -\lambda).$$

It is easy to check (see also [GSS88]) that $u_1 := x^2$, $u_2 := x\lambda$ and $u_3 := \lambda^2$ are generators of $\mathcal{E}_{x\lambda}(\mathbf{Z}_2)$ and that they satisfy the relation

$$u_1 u_3 - u_2^2 = 0.$$

Similarly one finds that $\overline{\mathcal{E}}_{x\lambda}(\mathbf{Z}_2)$ is generated by x and λ over $\mathcal{E}_{x\lambda}(\mathbf{Z}_2)$ and $\overline{\mathcal{E}}_{x\lambda}(\mathbf{Z}_2) = \mathcal{E}_{x\lambda}(\mathbf{Z}_2)$. Hence the extended $\mathbf{Z}_2$-tangent space is

$$T_e^{\mathbf{Z}_2}(g) = \mathcal{E}_{u_1 u_2 u_3}\{g, xg_x, \lambda g_x\} + \mathcal{E}_{u_3}\{\lambda g_\lambda\}. \tag{3.1}$$

The extended tangent space in the corresponding case without symmetry (i. e. $\Gamma = 1$), which is studied in [GS84, Key86] is

$$T^1_e(g) = \mathcal{E}_{x\lambda}\{g, g_x\} + \mathcal{E}_\lambda\{g_\lambda\} .$$

Since

$$T^{\mathbf{Z}_2}_e(g) = T^1_e(g) \cap \vec{\mathcal{E}}_{x\lambda}(\Gamma),$$

it follows that any $\mathbf{Z}_2$-equivariant bifurcation problem g, which is finitely-determined with respect to 1-equivalence, is finitely-determined with respect to $\mathbf{Z}_2$-equivalence as well.

Example 2.3.1 Let $g = \varepsilon x^3 - \lambda$, where $\varepsilon \in \{-1, +1\}$. This is called a hysteresis bifurcation in [GS84]. Calculating $T^{\mathbf{Z}_2}_e(g)$ yields

$$\begin{aligned} T^{\mathbf{Z}_2}_e(g) &= \mathcal{E}_{u_1 u_2 u_3}\left\{\varepsilon x^3 - \lambda, 3\varepsilon x^3, 3\varepsilon x^2\lambda\right\} + \mathcal{E}_{u_3}\{-\lambda\} \\ &= \mathcal{E}_{u_1 u_2 u_3}\{x^3, \lambda, x^2\lambda\} . \end{aligned}$$

It follows that

$$\frac{\vec{\mathcal{E}}_{x\lambda}(\mathbf{Z}_2)}{T^{\mathbf{Z}_2}_e(g)} = \mathbf{R}\{x\} .$$

This implies that $\operatorname{codim}^{\mathbf{Z}_2}(g) = 1$ and a muniversal unfolding of g is given by $\varepsilon x^3 - \lambda + \alpha x$.

It is easy to check that $g = \varepsilon x^3 - \lambda$ is the bifurcation of least possible $\mathbf{Z}_2$-codimension and hence can be regarded as the generic $\mathbf{Z}_2$-equivariant bifurcation: Consider first a bifurcation $h \in \vec{\mathcal{E}}_{x\lambda}(\mathbf{Z}_2)$ having a non-vanishing 1-jet. It follows by Mather's lemma that h is in either of the two orbits in the space of 1-jets represented by λ and $\lambda - \varepsilon x^3$. Continuing this reasoning one finds a family of $\mathbf{Z}_2$-equivariant germs

$$g_m = \varepsilon x^{2m+1} - \lambda \quad \text{for } m \geq 1 ,$$

where

$$\frac{\vec{\mathcal{E}}_{x\lambda}(\mathbf{Z}_2)}{T^{\mathbf{Z}_2}_e(g_m)} = \mathbf{R}\left\{x^{2k+1} : 0 \leq k \leq m-1\right\}$$

so that $\operatorname{codim}^{\mathbf{Z}_2}(g_m) = m$. This result corresponds to the family $g_m = \varepsilon x^k + \delta\lambda$ (where $\delta \in \{-1, +1\}$)[1] in the unsymmetric case. (Compare [GS84].) The complement of $T_e^{\mathbf{Z}_2}(g_m)$ in $\vec{\mathcal{E}}_{x\lambda}(\mathbf{Z}_2)$ can — as one would expect — be obtained by removing all the terms x^l for even l from the complement of $T_e^{\mathbf{1}}(g_m)$ in $\vec{\mathcal{E}}_{x\lambda}(\mathbf{1}) = \mathcal{E}_{x\lambda}$. These are precisely those terms which are not $\mathbf{Z}_2$-equivariant. Now consider a bifurcation $h \in \vec{\mathcal{E}}_{x\lambda}(\mathbf{Z}_2)$ with vanishing 1-jet. It follows by formula 3.1 that $\operatorname{codim}^{\mathbf{Z}_2}(h) \geq 2$.

3 $\mathbf{D}_4$-symmetry

3.1 Introduction

In this section the following action of $\mathbf{D}_4$ on $\mathbf{R}^2 \times \mathbf{R}^2$ generated by

$$\kappa.(x_1, x_2, \lambda_1, \lambda_2) := (x_1, -x_2, \lambda_1, \lambda_2)$$

and

$$\mu.(x_1, x_2, \lambda_1, \lambda_2) := (x_2, x_1, \lambda_1, -\lambda_2)$$

will be studied. This is motivated by a problem in mechanics. Consider a thin square plate and two pairs of forces F_1 and F_2 acting on it horizontally and vertically as shown in figure 1.1. Constructing a mathematical model for this situation leads to a description where two coordinates x_1, x_2 represent different buckling modes and two parameters F_1, F_2 give the values of the forces. To study physically interesting phenomena like buckling of the plate one can try to exploit the fact that the model has a certain symmetry: The physical situation does not change under the transformations

$$(x_1, x_2, F_1, F_2) \longrightarrow (x_1, -x_2, F_1, F_2)$$

and

$$(x_1, x_2, F_1, F_2) \longrightarrow (x_2, x_1, F_2, F_1)$$

[1] The sign δ does not appear in [GS84], since there the condition $A_\lambda(0) > 0$ is used in the definition of equivalence.

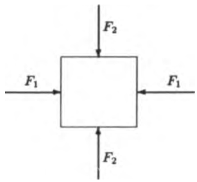

Figure 1.1: Forces acting on a square plate

These two transformations generate a $\mathbf{D}_4$-action. Introducing the new variables $\lambda_1 := F_1 + F_2$ and $\lambda_2 := F_1 - F_2$ this action appears in the form given above.

The following is an outline of the material contained in this section. from now on Γ will always refer to the particular action of $\mathbf{D}_4$ just described. The subsections 3.2 and 3.3 give results which explicitly describe the ring $\mathcal{E}_{x\lambda}(\Gamma)$ of Γ-invariant function germs, the module $\vec{\mathcal{E}}_{x\lambda}(\Gamma)$ of Γ-equivariant map germs and the module $\overleftrightarrow{\mathcal{E}}_{x\lambda}(\Gamma)$ of Γ-equivariant matrix-valued germs. For the latter two sets of generators are found, which generate these modules freely over a ring $\mathcal{E}_{u_1,\ldots,u_4}$, where $u_1, \ldots, u_4$ are certain Γ-invariant germs. Using these results the tangent spaces $T_e^\Gamma(g)$ and $T^\Gamma(g, U)$ are determined in terms of the invariants and equivariants in subsection 3.4. This is the most convenient way of doing calculations involving $T_e^\Gamma(g)$ and $T^\Gamma(g, U)$. Subsection 3.5 contains the main result, namely a normal form for generic

$\mathbf{D}_4$-equivariant bifurcations. it is obtained by using theorem 2.2.11 to determine the module of higher-order terms for the normal form. to this end it is necessary to work out $T^{\Gamma}(g,U)$ explicitly. this proves to be rather complicated and involves using the Mather-Malgrange preparation theorem (see [Mar82] and also part one). The next step is to show that $T^{\Gamma}(g,U)$ is intrinsic in the sense of definition 2.2.12. Finally the normal form is obtained by a scaling transformation. The last subsection contains bifurcation diagrams for the normal form.

3.2 Invariants

Let Γ denote the action of $\mathbf{D}_4$ on $\mathbf{R}^2 \times \mathbf{R}^2$ generated by

$$\kappa.(x_1,x_2,\lambda_1,\lambda_2) := (x_1,-x_2,\lambda_1,\lambda_2)$$

and

$$\mu.(x_1,x_2,\lambda_1,\lambda_2) := (x_2,x_1,\lambda_1,-\lambda_2).$$

Let N,δ,Δ,u_4 denote the following expressions

$$\begin{aligned} N &:= x_1^2+x_2^2 \\ \delta &:= x_2^2-x_1^2 \\ \Delta &:= \delta^2 \\ u_4 &:= \lambda_2^2. \end{aligned}$$

Also let $u := (u_1,\ldots,u_5)$.

Proposition 3.2.1 *The ring $\mathcal{E}_{x\lambda}(\Gamma)$ of smooth Γ-invariant functions can be written as $\mathcal{E}_u$, where $u_1 = N$, $u_2 = \Delta$, $u_3 = \lambda_1$, $u_4 = \lambda_2^2$, $u_5 = \delta\lambda_2$.*

Proof. By a theorem of Schwarz [Sch75] it is sufficient to show that every Γ-invariant polynomial can be written as a polynomial in u.

Let f be a polynomial in $\mathbf{R}[x_1,x_2,\lambda_1,\lambda_2]$. Assume that f is Γ-invariant. This is equivalent to the following two conditions:

$$f(x_1,x_2,\lambda_1,\lambda_2) = f(x_1,-x_2,\lambda_1,\lambda_2) \tag{2.1}$$

and

$$f(x_1, x_2, \lambda_1, \lambda_2) = f(x_2, x_1, \lambda_1, -\lambda_2). \tag{2.2}$$

It follows from (2.1) that f is an even function of x_2. Hence there exists a polynomial $\overline{f} \in \mathbf{R}[x_1, x_2, \lambda_1, \lambda_2]$ such that

$$f = \overline{f}(x_1, x_2^2, \lambda_1, \lambda_2).$$

Using this and (2.2) it follows that

$$\overline{f}(x_1, x_2^2, \lambda_1, \lambda_2) = \overline{f}(x_2, x_1^2, \lambda_1, -\lambda_2). \tag{2.3}$$

The right hand side of this equation is an even function of x_1, hence the left hand side is as well. Therefore there exists a polynomial $\hat{f}$ such that

$$\overline{f} = \hat{f}(x_1^2, x_2^2, \lambda_1, \lambda_2)$$

and therefore

$$f(x_1, x_2, \lambda_1, \lambda_2) = \hat{f}(x_1^2, x_2^2, \lambda_1, \lambda_2).$$

The result of the action of μ on $(x_1^2, x_2^2, \lambda_1, \lambda_2)$ is $(x_2^2, x_1^2, \lambda_1, -\lambda_2)$. Equivalently the result of its action on $(N, \delta, \lambda_1, \lambda_2)$ is $(N, -\delta, \lambda_1, -\lambda_2)$. It is easy to see that the invariants for the $\mathbf{Z}_2$-action on $\mathbf{R}^2$ defined by

$$-1.(x, y) := (-x, -y)$$

are x^2, xy and y^2. Hence the invariants in the case above are N, λ_1 and $\delta^2 = \Delta$, $\delta\lambda_2$ and λ_2^2. It follows that f can be written as a polynomial in $u_1, \ldots, u_5$, which proves the result. □

Remark 3.2.2 The invariants $u_1, \ldots, u_5$ satisfy one relation, namely

$$u_5^2 = u_2 u_4.$$

Otherwise there are no relations.

3.3 Equivariants

Proposition 3.3.1 *$\vec{\mathcal{E}}_{x\lambda}(\Gamma)$ is generated by*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \delta\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} \quad \textit{and} \quad \lambda_2\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}$$

as an $\mathcal{E}_{x\lambda}(\Gamma)$-module.

Proof. By a theorem of Poenaru [Poe76] it is sufficient to show that

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \delta\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} \quad \text{and} \quad \lambda_2\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}$$

generate $\vec{\mathcal{P}}(\Gamma)$ — the $\mathcal{P}(\Gamma)$-module of Γ-equivariant polynomials. Let $f \in \vec{\mathcal{P}}(\Gamma)$. f can be witten as

$$f(x_1, x_2, \lambda_1, \lambda_2) = \begin{pmatrix} f_1(x_1, x_2, \lambda_1, \lambda_2) \\ f_2(x_1, x_2, \lambda_1, \lambda_2) \end{pmatrix},$$

where $f_1, f_2 \in \mathbf{R}[x_1, x_2, \lambda_1, \lambda_2]$.

Since f is Γ-equivariant, f commutes with κ. This is equivalent to

$$\begin{aligned} f_1(x_1, -x_2, \lambda_1, \lambda_2) &= f_1(x_1, x_2, \lambda_1, -\lambda_2) \\ f_2(x_1, -x_2, \lambda_1, \lambda_2) &= -f_2(x_1, x_2, \lambda_1, \lambda_2). \end{aligned}$$

It follows that f_1 is an even and f_2 an odd function of x_2. Hence there exist polynomials $\overline{f_1}$ and $\overline{f_2}$ such that

$$\begin{aligned} f_1(x_1, x_2, \lambda_1, \lambda_2) &= \overline{f_1}(x_1, x_2^2, \lambda_1, \lambda_2) \\ f_2(x_1, x_2, \lambda_1, \lambda_2) &= x_2\overline{f_2}(x_1, x_2^2, \lambda_1, \lambda_2). \end{aligned}$$

Using the fact that f commutes with μ as well a similar argument shows that

$$f(x_1, x_2, \lambda_1, \lambda_2) = \begin{pmatrix} x_1 \hat{f}_1(x_1^2, x_2^2, \lambda_1, \lambda_2) \\ x_2 \hat{f}_2(x_1^2, x_2^2, \lambda_1, \lambda_2) \end{pmatrix} \tag{3.1}$$

for some polynomials f_1 and f_2. Define $\alpha := x_1^2$ and $\beta := x_2^2$ and consider the mapping f given by

$$f(\alpha,\beta,\lambda_1,\lambda_2) = \begin{pmatrix} \hat{f}_1(\alpha,\beta,\lambda_1,\lambda_2) \\ \hat{f}_2(\alpha,\beta,\lambda_1,\lambda_2) \end{pmatrix}.$$

The condition $\mu.f = f\mu$ can be restated as

$$\begin{aligned} \hat{f}_1(\beta,\alpha,\lambda_1,-\lambda_2) &= \hat{f}_2(\alpha,\beta,\lambda_1,\lambda_2) && (3.2)\\ \hat{f}_2(\beta,\alpha,\lambda_1,-\lambda_2) &= \hat{f}_1(\alpha,\beta,\lambda_1,\lambda_2). && (3.3) \end{aligned}$$

Defining

$$g_1 := \frac{1}{2}(\hat{f}_1 + \hat{f}_2)$$

and

$$g_2 := \frac{1}{2}(\hat{f}_1 - \hat{f}_2)$$

the last two equations are equivalent to $g_1\mu = g_1$ and $g_2\mu = -g_2$. Since κ acts trivially on $(\alpha,\beta,\lambda_1,\lambda_2)$ the first condition means that g_1 is Γ-invariant.

The second condition implies that

$$g_2 = g_{21}(\alpha - \beta) + g_{22}\lambda_2,$$

which can be seen by considering the $\mathbf{Z}_2$-action mentioned in the proof of proposition (3.2.1). Returning to (3.1) resubstituting yields

$$f_1(x_1,x_2,\lambda_1,\lambda_2) = g_1\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - g_{21}\delta\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} + g_{22}\lambda_2\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix},$$

which proves the result. $\square$

Proposition 3.3.2 *$\overline{\mathcal{E}}_{x\lambda}(\Gamma)$ is generated by*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \lambda_2 & 0 \\ 0 & -\lambda_2 \end{pmatrix}, \begin{pmatrix} \delta & 0 \\ 0 & -\delta \end{pmatrix},$$

$$\begin{pmatrix} 0 & x_1x_2 \\ x_1x_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_1x_2\lambda_2 \\ -x_1x_2\lambda_2 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & x_1x_2\delta \\ -x_1x_2\delta & 0 \end{pmatrix}$$

as an $\mathcal{E}_{x\lambda}(\Gamma)$-module.

Proof. This proof is very similar to the one of proposition 3.3.1. For this reason some of the details are omitted. Let $S \in \overline{\mathcal{E}}_{x\lambda}(\Gamma)$. This is equivalent to the two conditions

$$S(\kappa.(x_1, x_2, \lambda_1, \lambda_2)) = \kappa.S(x_1, x_2, \lambda_1, \lambda_2)\kappa^{-1}$$

and

$$S(\mu.(x_1, x_2, \lambda_1, \lambda_2)) = \mu.S(x_1, x_2, \lambda_1, \lambda_2)\mu^{-1}.$$

As before we can assume that the components of S are polynomials. Using the definitions for κ and μ it can be shown that there exist polynomials $\hat{s}_{ij}$ $(1 \le i, j \le 2)$ such that

$$S(x_1, x_2, \lambda_1, \lambda_2) = \begin{pmatrix} \hat{s}_{11}(x_1^2, x_2^2, \lambda_1, \lambda_2) & x_1x_2\hat{s}_{12}(x_1^2, x_2^2, \lambda_1, \lambda_2) \\ x_1x_2\hat{s}_{12}(x_2^2, x_1^2, \lambda_1, -\lambda_2) & \hat{s}_{11}(x_2^2, x_1^2, \lambda_1, -\lambda_2) \end{pmatrix} \tag{3.4}$$

The following polynomial mappings ($\alpha := x_1^2$, $\beta := x_2^2$)

$$S_1(\alpha, \beta, \lambda_1, \lambda_2) = \begin{pmatrix} \hat{s}_{11}(\alpha, \beta, \lambda_1, \lambda_2) \\ \hat{s}_{11}(\beta, \alpha, \lambda_1, -\lambda_2) \end{pmatrix}$$

and

$$S_2(\alpha, \beta, \lambda_1, \lambda_2) = \begin{pmatrix} \hat{s}_{12}(\alpha, \beta, \lambda_1, \lambda_2) \\ \hat{s}_{12}(\beta, \alpha, \lambda_1, -\lambda_2) \end{pmatrix}$$

both satisfy conditions (3.2) and (3.3) in the proof of proposition 3.2.1. The reasoning there shows that the diagonal elements of the matrix in (3.4) are of the types

$$a\begin{pmatrix} 1 \\ 1 \end{pmatrix}, b\begin{pmatrix} \lambda_2 \\ -\lambda_2 \end{pmatrix}, c\begin{pmatrix} \delta \\ -\delta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $a, b, c \in \mathcal{E}_{x\lambda}(\Gamma)$. The last case occurs, when $S_1 = 0$.

In the same way it follows that the off-diagonal elements in 3.4 are of the types

$$e\begin{pmatrix} x_1x_2 \\ x_1x_2 \end{pmatrix}, f\begin{pmatrix} x_1x_2\lambda_2 \\ -x_1x_2\lambda_2 \end{pmatrix}, g\begin{pmatrix} x_1x_2\delta \\ -x_1x_2\delta \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $e, f, g \in \mathcal{E}_{x\lambda}(\Gamma)$. The result follows. □

Proposition 3.3.3 *A)* $\mathcal{E}_\lambda(\Gamma) = \mathcal{E}_{\lambda_1, u_4}$, *where* $u_4 = \lambda_2^2$.
B) $\vec{\mathcal{E}}_\lambda(\Gamma)$ *is freely generated by*

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ \lambda_2 \end{pmatrix}$$

as an $\mathcal{E}_\lambda(\Gamma)$*-module.*

Proof. A) This follows immediately from proposition 3.2.1.
B) Let

$$\Lambda(\lambda_1, \lambda_2) = \begin{pmatrix} \Lambda_1(\lambda_1, \lambda_2) \\ \Lambda_2(\lambda_1, \lambda_2) \end{pmatrix}$$

be Γ-equivariant. This is equivalent to $\mu.\Lambda = \Lambda\mu$, i. e.

$$\begin{pmatrix} \Lambda_1(\lambda_1, \lambda_2) \\ -\Lambda_2(\lambda_1, \lambda_2) \end{pmatrix} = \begin{pmatrix} \Lambda_1(\lambda_1, -\lambda_2) \\ \Lambda_2(\lambda_1, -\lambda_2) \end{pmatrix}.$$

Hence Λ_1 is an even and Λ_2 an odd function of λ_2, which implies that $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \lambda_2 \end{pmatrix}$ generate $\vec{\mathcal{E}}_\lambda(\Gamma)$ over $\mathcal{E}_\lambda(\Gamma)$. It is straightforward to check that these generators are free. □

Unlike as for $\vec{\mathcal{E}}_\lambda(\Gamma)$ the sets of generators given for $\vec{\mathcal{E}}_{x\lambda}(\Gamma)$ and $\vec{\mathcal{E}}_{x\lambda}(\Gamma)$ are not free. This is due to the fact that the Γ- invariants $u_1, \ldots, u_5$ satisfy a relation. (See remark 3.2.2.) However, for the tangent space calculations in subsection 3.4 it will be advantageous to work with free modules. To this end we show that both $\vec{\mathcal{E}}_{x\lambda}(\Gamma)$ and $\vec{\mathcal{E}}_{x\lambda}(\Gamma)$ can be written as free modules over $\mathcal{E}_{u_1, \ldots, u_4}$ by increasing the number of generators.

Proposition 3.3.4 $\vec{\mathcal{E}}_{x\lambda}(\Gamma)$ *is freely generated by*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \delta \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}, \lambda_2 \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}, \delta\lambda_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

as an $\mathcal{E}_{u_1, \ldots, u_4}$*-module.*

Proof. Let $g \in \vec{\mathcal{E}}_{x\lambda}(\Gamma)$. By proposition 3.3.1 g can be written as

$$g = p\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + q\delta\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} + r\lambda_2\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}$$

where $p, q, r \in \mathcal{E}_{x\lambda}(\Gamma) = \mathcal{E}_{u_1,\dots,u_4}$. Since $u_3^2 = u_2u_4$, there exist germs $p_i, q_i, r_i \in \mathcal{E}_{u_1,\dots,u_4}$ $(i = 1, 2)$ such that

$$p = p_1 + \delta\lambda_2 p_2\,,$$

$$q = q_1 + \delta\lambda_2 q_2\,,$$

$$r = r_1 + \delta\lambda_2 r_2\,,$$

Hence

$$\begin{aligned} p\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= p_1\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + p_2\delta\lambda_2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \\ q\delta\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} &= q_1\delta\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} + \Delta q_2\lambda_2\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}, \\ r\lambda_2\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} &= r_1\lambda_2\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} + u_4 r_2\delta\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}. \end{aligned}$$

It follows that

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \delta\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}, \lambda_2\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}, \delta\lambda_2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

generate $\vec{\mathcal{E}}_{x\lambda}(\Gamma)$ as an $\mathcal{E}_{u_1,\dots,u_4}$-module.

Now suppose

$$a\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + b\delta\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} + c\lambda_2\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} + d\delta\lambda_2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $a, b, c, d \in \mathcal{E}_{u_1,\dots,u_4}$. This is equivalent to two equations:

$$a + b\delta + c\lambda_2 + d\delta\lambda_2 = 0 \qquad (3.5)$$

$$a - b\delta - c\lambda_2 + d\delta\lambda_2 = 0\,. \qquad (3.6)$$

Adding these yields

$$2a + 2d\delta\lambda_2 = 0\,,$$

which is equivalent to

$$a = -\delta\lambda_2 d\,.$$

a is an even function of λ_2. This implies $d = 0$ and hence $a = 0$. Similarly, subtracting (3.6) from (3.5) yields

$$2b\delta + 2c\lambda_2 = 0\,,$$

which implies $c = 0$ and $b = 0$. Hence the four mappings given above are free generators of $\overline{\mathcal{E}}_{x\lambda}(\Gamma)$ over $\mathcal{E}_{u_1,\ldots,u_4}$. □

Proposition 3.3.5 *$\overline{\mathcal{E}}_{x\lambda}(\Gamma)$ is freely generated by*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \lambda_2 & 0 \\ 0 & -\lambda_2 \end{pmatrix}, \begin{pmatrix} \delta & 0 \\ 0 & -\delta \end{pmatrix},$$

$$\begin{pmatrix} 0 & x_1x_2 \\ x_1x_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_1x_2\lambda_2 \\ -x_1x_2\lambda_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_1x_2\delta \\ -x_1x_2\delta & 0 \end{pmatrix},$$

$$\delta\lambda_2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \delta\lambda_2\begin{pmatrix} 0 & x_1x_2 \\ x_1x_2 & 0 \end{pmatrix}$$

as an $\mathcal{E}_{u_1,\ldots,u_4}$-module.

Proof. Let $S_1, \ldots, S_6$ be the matrices listed in proposition 3.3.2 and let

$$\begin{aligned} S_7 &= \delta\lambda_2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ S_8 &= \delta\lambda_2\begin{pmatrix} 0 & x_1x_2 \\ x_1x_2 & 0 \end{pmatrix}. \end{aligned}$$

Let $F = \mathcal{E}_{u_1,\ldots,u_4}\{S_1, \ldots, S_8\}$. To show that $F = \overline{\mathcal{E}}_{x\lambda}(\Gamma)$, it is sufficient to check that $\delta\lambda_2 S_j \in F$ for $j = 1, \ldots, 6$. This condition is shown to hold by the following calculations:

$$\begin{aligned}
\delta\lambda_2 S_1 &= S_7 , \\
\delta\lambda_2 S_2 &= u_4 S_3 , \\
\delta\lambda_2 S_3 &= \Delta S_2 , \\
\delta\lambda_2 S_4 &= S_8 , \\
\delta\lambda_2 S_5 &= u_4 S_6 , \\
\delta\lambda_2 S_6 &= \Delta S_5 .
\end{aligned}$$

Now suppose

$$\sum_{i=1}^{8} a_i S_i = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} ,$$

where $a_i \in \mathcal{E}_{u_1,\dots,u_4}$ for $i = 1,\dots,8$. This condition is equivalent to four equations involving $a_1,\dots,a_8$. It is easy to check that these equations consist of two pairs each of which can be treated analogously to the proof of proposition 3.3.4. In this way it follows that $a_i = 0$ for $i = 1,\dots,8$ and hence $S_1,\dots,S_8$ are free generators of $\overrightarrow{\mathcal{E}}_{x\lambda}(\Gamma)$ over $\mathcal{E}_{u_1,\dots,u_4}$. □

3.4 Tangent spaces for the D_4-action

To be able to conveniently calculate with Γ-equivariant germs, we use invariant notation. Compare [GSS88, GR87] . From now on the abbreviation $\overline{u} := (u_1,\dots,u_4)$ is used.

Definition 3.4.1 A) Let $p, q, r, s \in \mathcal{E}_{\overline{u}}$. Then

$$[p,q,r,s] := p\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + q\delta\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} + r\lambda_2\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} + s\delta\lambda_2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} .$$

B) Let I_i be ideals in $\mathcal{E}_{\overline{u}}$ for $i = 1,\dots,4$. Then we define the following submodule of $\overrightarrow{\mathcal{E}}_{x\lambda}(\Gamma)$:

$$[I_1, I_2, I_3, I_4] := I_1[1,0,0,0] + I_2[0,1,0,0] + I_3[0,0,1,0] + I_4[0,0,0,1] .$$

Remark 3.4.2 By proposition 3.3.4 each germ $g \in \overrightarrow{\mathcal{E}}_{x\lambda}(\Gamma)$ can be written uniquely as $g = [p,q,r,s]$ for some $p,q,r,s \in \mathcal{E}_{\overline{u}}$. In other words $\overrightarrow{\mathcal{E}}_{x\lambda}(\Gamma)$ can be identified with $\mathcal{E}_{\overline{u}} \oplus \mathcal{E}_{\overline{u}} \oplus \mathcal{E}_{\overline{u}} \oplus \mathcal{E}_{\overline{u}}$.

The remainder of this subsection is devoted to formulae for the various tangent spaces.

Proposition 3.4.3 *Let $g = [p, q, r, s]$, where $p, q, r, s \in \mathcal{E}_{\mathbf{u}}$. Then*

$$T_e(g) = \mathcal{E}_{\mathbf{u}}\{g_1, \ldots, g_{12}\} + \mathcal{E}_{\lambda_1 u_4}\{g_{13}, g_{14}\},$$

where

$$\begin{aligned}
g_1 &= [p, q, r, s], \\
g_2 &= [u_4 r, u_4 s, p, q], \\
g_3 &= [\Delta q, p, \Delta s, r], \\
g_4 &= [Np - \Delta q, p - Nq, -Nr + \Delta s, -r + Ns], \\
g_5 &= [\Delta p - N\Delta q, Np - \Delta q, N\Delta s - \Delta r, \Delta s - Nr], \\
g_6 &= [-Nu_4 r + \Delta u_4 s, -u_4 r + Nu_4 s, Np - \Delta q, p - Nq], \\
g_7 &= [\Delta u_4 s, u_4 r, \Delta q, p], \\
g_8 &= [-\Delta u_4 r + N\Delta u_4 s, -Nu_4 r + \Delta u_4 s, \Delta p - N\Delta q, Np - \Delta q], \\
g_9 &= [2Np_N + 4\Delta p_\Delta + p, 2Nq_N + 4\Delta q_\Delta + 3q, 2Nr_N + 4\Delta r_\Delta + r, \\
&\qquad 2Ns_N + 4\Delta s_\Delta + 3s], \\
g_{10} &= [-2\Delta p_N - 4N\Delta p_\Delta + \Delta q, -2\Delta q_N - 4N\Delta q_\Delta - 2Nq + p, \\
&\qquad -2\Delta r_N - 4N\Delta r_\Delta + \Delta s, -2\Delta s_N - 4N\Delta s_\Delta - 2Ns + r], \\
g_{11} &= [-2\Delta u_4 s_N - 4N\Delta u_4 s_\Delta - 2Nu_4 s + u_4 r, -2u_4 q_N - 4Nu_4 q_\Delta + u_4 s, \\
&\qquad -2\Delta q_N - 4N\Delta q_\Delta - 2qN + p, -2p_N - 4Np_\Delta + q], \\
g_{12} &= [2N\Delta u_4(p_N + s_N) + 4\Delta^2 u_4(p_\Delta + s_\Delta) + 3\Delta u_4 s, \\
&\qquad 2Nu_4 r_N + 4\Delta u_4 r_\Delta + u_4 r, 2N\Delta q_N + 4\Delta^2 q_\Delta + 3\Delta q, p], \\
g_{13} &= [p_{\lambda_1}, q_{\lambda_1}, r_{\lambda_1}, s_{\lambda_1}], \\
g_{14} &= [2u_4 p_{u_4}, 2u_4 q_{u_4}, 2u_4 r_{u_4} + r, 2u_4 s_{u_4} + s].
\end{aligned}$$

Proof. By definition 2.2.6

$$T_e(g) = \overleftrightarrow{\mathcal{E}}_{x\lambda}(\Gamma)g + (D_x g)\overline{\mathcal{E}}_{x\lambda}(\Gamma) + (D_\lambda g)\overline{\mathcal{E}}_\lambda(\Gamma).$$

The generators $g_1, \ldots, g_8$ are obtained by proposition 3.3.5. We have

$$g_i = S_i[p, q, r, s]$$

for $i = 1, \ldots, 8$, where S_i are the generators of $\overrightarrow{\mathcal{E}}_{x\lambda}(\Gamma)$ from proposition 3.3.5. Table 4.1 displays a list of all the products $S_i y_j$ ($i = 1, \ldots, 8$ $j = 1, \ldots, 4$), where y_j are the generators of $\overrightarrow{\mathcal{E}}_{x\lambda}(\Gamma)$ from proposition 3.3.4. The

	y_1	y_2	y_3	y_4
S_1	$[1,0,0,0]$	$[0,1,0,0]$	$[0,0,1,0]$	$[0,0,0,1]$
S_2	$[0,0,1,0]$	$[0,0,0,1]$	$[u_4,0,0,0]$	$[0,u_4,0,0]$
S_3	$[0,1,0,0]$	$[\Delta,0,0,0]$	$[0,0,0,1]$	$[0,0,\Delta,0]$
S_4	$\frac{1}{2}[N,1,0,0]$	$-\frac{1}{2}[\Delta,N,0,0]$	$-\frac{1}{2}[0,0,N,1]$	$\frac{1}{2}[0,0,\Delta,N]$
S_5	$\frac{1}{2}[\Delta,N,0,0]$	$-\frac{1}{2}[N\Delta,\Delta,0,0]$	$-\frac{1}{2}[0,0,\Delta,N]$	$\frac{1}{2}[0,0,N\Delta,\Delta]$
S_6	$\frac{1}{2}[0,0,N,1]$	$-\frac{1}{2}[0,0,\Delta,N]$	$-\frac{1}{2}[Nu_4,u_4,0,0]$	$\frac{1}{2}[\Delta u_4,Nu_4,0,0]$
S_7	$[0,0,0,1]$	$[0,0,\Delta,0]$	$[0,u_4,0,0]$	$[\Delta u_4,0,0,0]$
S_8	$\frac{1}{2}[0,0,\Delta,N]$	$-\frac{1}{2}[0,0,N\Delta,\Delta]$	$-\frac{1}{2}[\Delta u_4,Nu_4,0,0]$	$\frac{1}{2}[N\Delta u_4,\Delta u_4,0,0]$

Table 4.1: The products $S_i y_j$

expressions for $g_1, \ldots, g_8$ follow immediately from table 4.1. (Table 4.1 will be used again in the proof of lemma 3.5.7.)

The generators $g_9, \ldots, g_{12}$ arise from $(D_x g)\overrightarrow{\mathcal{E}}_{x\lambda}(\Gamma)$. They are obtained as follows. We have

$$g_j = (D_x g) y_j$$

for $j = 1, \ldots, 4$. Note that

$$D_x g = D_x\left(p\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) + D_x\left(q\delta\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}\right) + D_x\left(r\lambda_2\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}\right)$$

$$+ D_x\left(s\delta\lambda_2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right). \tag{4.1}$$

The formulae for $g_9, \ldots, g_{12}$ now follow by calculating all the products of the four terms in (4.1) and $y_1, \ldots, y_4$. One sample calculation will be given for

$$D_x\left(q\delta\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}\right)\lambda_2\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}.$$

Firstly, we have

$$\left(\frac{\partial q}{\partial x_1}, \frac{\partial q}{\partial x_2}\right) = (2x_1q_N - 4x_1\delta q_\Delta, 2x_2q_N + 4x_2\delta q_\Delta).$$

Hence

$$\begin{aligned} D_x\left(q\delta\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}\right) &= \delta\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}(2x_1q_N - 4x_1\delta q_\Delta, 2x_2q_N + 4x_2\delta q_\Delta) \\ &\quad + qD_x\left(\delta\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}\right) \\ &= \delta\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}(2x_1q_N - 4x_1\delta q_\Delta, 2x_2q_N + 4x_2\delta q_\Delta) \\ &\quad + q\begin{pmatrix} -2x_1^2 + \delta & 2x_1x_2 \\ 2x_1x_2 & -2x_2^2 - \delta \end{pmatrix}. \end{aligned}$$

This yields

$$\begin{aligned} D_x\left(q\delta\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}\right)\lambda_2\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} &= \delta\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}(-2q_N - 4Nq_\Delta)\delta\lambda_2 \\ &\quad + q\lambda_2\begin{pmatrix} x_1(\delta - 2N) \\ x_2(\delta + 2N) \end{pmatrix} \\ &= [0, 0, -2\Delta q_N - 4N\Delta q_\Delta - 2qN, q]. \end{aligned}$$

The calculations for the generators g_{13} and g_{14} arising from $(D_\lambda g)\overline{\mathcal{E}}_\lambda(\Gamma)$ are considerably easier and are therefore omitted. □

Proposition 3.4.4 *Let $g = [p, q, r, s]$, where $p, q, r, s \in \mathcal{E}_{\mathbf{u}}$ and let $g_1, \ldots, g_{14}$ be as in proposition 3.4.3. Then*

$$\begin{aligned} T(g, U) \;=\; & \mathcal{E}_{\mathbf{u}}\{Ng_1, \Delta g_1, \lambda_1 g_1, u_4 g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8, Ng_9, \Delta g_9, \lambda_1 g_9, \\ & \quad u_4 g_9, g_{10}, g_{11}, g_{12}\} \\ & + \mathcal{E}_{\lambda_1 u_4}\{\lambda_1 g_{13}, u_4 g_{13}, \lambda_1 g_{14}, u_4 g_{14}\} . \end{aligned}$$

Proof. The result follows in the same way as for $T_e(g)$ — the only difference being the use of the conditions for unipotent $\mathbf{D}_4$-equivalences. (Compare definition 2.2.9.) □

Proposition 3.4.5 *Let $g = [p, q, r, s]$, where $p, q, r, s \in \mathcal{E}_u$ and let $g_1, \ldots, g_{14}$ be as in proposition 3.4.3. Then*

$$T_e(g) = T(g, U) + \mathbf{R}\{g_1, g_9, g_{13}, \lambda_1 g_{13}, g_{14}\} .$$

Proof. This follows immediately from propositions 3.4.3 and 3.4.4. □

3.5 The generic normal form

Theorem 3.5.1 *Let $g = [\epsilon_0 \lambda_1 + \alpha N, \epsilon_1, 1, 0]$, where $\epsilon_0, \epsilon_1 \in \{-1, +1\}$ and $\alpha \neq 0, \epsilon_1$. Then g has $\mathbf{D}_4$-codimension 1. All bifurcations $h = [p, q, r, s]$, which satisfy $p = 0, q \neq 0$, $p_N \neq 0$, $p_N - q \neq 0$, $p_{\lambda_1} \neq 0$ and $r \neq 0$ are equivalent to g, with the coefficients ϵ_0, ϵ_1 and α satisfying the conditions $\epsilon_0 = \operatorname{sg} p_{\lambda_1}$, $\epsilon_1 = \operatorname{sg} q$ and $\alpha = p_N/|q|$. Bifurcations which do not satisfy all non-degeneracy conditions are of higher codimension than g.*

The proof of theorem 3.5.1 will be given at the end of this subsection.

Remark 3.5.2 The theorem can be interpreted in the following way: The parameter α which satisfies $\alpha = p_N/|q|$ can be regarded as a modulus. In the proof of theorem 3.5.1 it will be shown that

$$\begin{aligned} T_e(g) \;=\; & \left[< N^2, N\lambda_1, \lambda_1^2, \Delta, u_4 >, \mathcal{M}, \mathcal{M}, \mathcal{E}\right] \\ & + \mathbf{R}\left\{\alpha[N, 0, 0, 0] + \epsilon_1[0, 1, 0, 0], [1, 0, 0, 0], [\lambda_1, 0, 0, 0], [0, 0, 1, 0]\right\} . \end{aligned}$$

Here $\mathcal{M}$ denotes the maximal ideal in $\mathcal{E}_{\mathbf{u}}$. Hence $[N,0,0,0]$ can be chosen as an unfolding term since

$$T_e(g) + \mathbf{R}[N,0,0,0] = \vec{\mathcal{E}}_{x\lambda}(\Gamma).$$

For this reason it is justified to regard the normal form $g = [\epsilon_0\lambda_1 + \alpha N, \epsilon_1, 1, 0]$ as the generic $\mathbf{D}_4$-equivariant bifurcation.

Lemma 3.5.3 *Let $g = [\lambda_1 + \alpha N, \beta, 1, 0]$, where $\alpha, \beta \neq 0$ and $\alpha \neq \beta$. Then*

$$T(g,U) = \left[< N^2, N\lambda_1, \lambda_1^2, \Delta, u_4 >, \mathcal{M}, \mathcal{M}, \mathcal{E}\right].$$

Proof. Applying proposition 3.4.4 with $p = \lambda_1 + \alpha N, q = \beta$, $r = 1, s = 0$ yields the following for the generators of $T(g,U)$:

$$\begin{aligned}
Ng_1 &= [N\lambda_1 + \alpha N^2, \beta N, N, 0],\\
\Delta g_1 &= [\Delta\lambda_1 + \alpha N\Delta, \beta\Delta, \Delta, 0],\\
\lambda_1 g_1 &= [\lambda_1^2 + \alpha N\lambda_1, \beta\lambda_1, \lambda_1, 0],\\
u_4 g_1 &= [\lambda_1 u_4 + \alpha N u_4, \beta u_4, u_4, 0],\\
g_2 &= [u_4, 0, \lambda_1 + \alpha N, \beta],\\
g_3 &= [\beta\Delta, \lambda_1 + \alpha N, 0, 1],\\
g_4 &= [N\lambda_1 + \alpha N^2 - \beta\Delta, \lambda_1 + (\alpha-\beta)N, -N, -1],\\
g_5 &= [\Delta\lambda_1 + (\alpha-\beta)N\Delta, N\lambda_1 + \alpha N^2 - \beta\Delta, -\Delta, -N],\\
g_6 &= [-Nu_4, -u_4, N\lambda_1 + \alpha N^2 - \beta\Delta, \lambda_1 + (\alpha-\beta)N],\\
g_7 &= [0, u_4, \beta\Delta, \lambda_1 + \alpha N],\\
g_8 &= [-\Delta u_4, -Nu_4, \Delta\lambda_1 + (\alpha-\beta)N\Delta, N\lambda_1 + \alpha N^2 - \beta\Delta],\\
Ng_9 &= [N\lambda_1 + 3\alpha N, N, 0],\\
\Delta g_9 &= [\Delta\lambda_1 + 3\alpha N\Delta, 3\beta\Delta, \Delta, 0],\\
\lambda_1 g_9 &= [\lambda_1^2 + 3\alpha N\lambda_1, 3\beta\lambda_1, \lambda_1, 0],\\
u_4 g_9 &= [\lambda_1 u_4 + 3\alpha N u_4, 3\beta u_4, u_4, 0],\\
g_{10} &= [(-2\alpha+\beta)\Delta, \lambda_1 + (\alpha-2\beta)N, 0, 1],\\
g_{11} &= [u_4, 0, \lambda_1 + (\alpha-2\beta)N, -2\alpha+\beta],
\end{aligned}$$

$$g_{12} = [2\alpha N\Delta u_4, u_4, 3\beta\Delta, \lambda_1 + \alpha N],$$

and

$$\begin{aligned} \lambda_1^2 g_{13} &= [\lambda_1^2, 0, 0, 0], \\ u_4 g_{13} &= [u_4, 0, 0, 0], \\ \lambda_1 g_{14} &= [0, 0, \lambda_1, 0], \\ u_4 g_{14} &= [0, 0, u_4, 0]. \end{aligned}$$

The proof is divided into three steps.

Step 1: *Calculation of the module*

$$\frac{\overline{\mathcal{E}}_u}{RT(g,U) + < \lambda_1, u_4 > \overline{\mathcal{E}}_{\overline{u}}}$$

It will be shown that

$$\frac{\overline{\mathcal{E}}_u}{RT(g,U) + < \lambda_1, u_4 > \overline{\mathcal{E}}_{\overline{u}}}$$

is generated by $[N,0,0,0]$,$[1,0,0,0]$,$[0,1,0,0]$ and $[0,0,1,0]$ as a vector space over $\mathbf{R}$ provided $\alpha, \beta \neq 0$ and $\alpha \neq \beta$. To this end consider the module

$$M := \mathcal{E}_{N\Delta}\{h_1, \ldots, h_{14}\},$$

where

$$\begin{aligned} h_1 &= [\alpha N^2, \beta N, N, 0], \\ h_2 &= [\alpha N\Delta, \beta\Delta, \Delta, 0], \\ h_3 &= [0, 0, \alpha N, \beta], \\ h_4 &= [\beta\Delta, \alpha N, 0, 1], \\ h_5 &= [\alpha N^2 - \beta\Delta, (\alpha - \beta)N, -N, -1], \\ h_6 &= [(\alpha - \beta)N\Delta, \alpha N^2 - \beta\Delta, -\Delta, -N], \\ h_7 &= [0, 0, \alpha N^2 - \beta\Delta, (\alpha - \beta)N], \end{aligned}$$

$$
\begin{aligned}
h_8 &= [0,0,\beta\Delta,\alpha N]\,,\\
h_9 &= [0,0,(\alpha-\beta)N\Delta,\alpha N^2-\beta\Delta]\,,\\
h_{10} &= [3\alpha N^2,3\beta N,N,0]\,,\\
h_{11} &= [3\alpha N\Delta,3\beta\Delta,\Delta,0]\,,\\
h_{12} &= [(-2\alpha+\beta)\Delta,(\alpha-2\beta)N,0,1]\,,\\
h_{13} &= [0,0,(\alpha-2\beta)N,-2\alpha+\beta]\,,\\
h_{14} &= [0,0,3\beta\Delta,\alpha N]\,.
\end{aligned}
$$

The generators h_i are obtained from the generators of $RT(g,U)$ working modulo $<\lambda_1,u_4>\overline{\mathcal{E}}_{\overline{u}}$. A straightforward — if cumbersome — calculation shows that

$$
M = \left[<N^2,\Delta>_{N\Delta}, \mathcal{M}_{N\Delta}, \mathcal{M}_{N\Delta}, \mathcal{E}_{N\Delta} \right]\,,
$$

if $\alpha,\beta \neq 0$ and $\alpha \neq \beta$. (Here the subscript $N\Delta$ indicates that all ideals are to be taken in the ring $\mathcal{E}_{N\Delta}$). The conditions for α and β arise in the following way: The calculation yields the terms

$$
\begin{aligned}
&\alpha^2N^2-\beta^2\Delta\\
&2\alpha(\beta-\alpha)N^2\\
&\alpha N^2-\alpha\Delta\\
&\alpha^2(2\alpha-\beta)N^2+(-2\alpha+\beta)\beta^2\Delta
\end{aligned}
$$

generating the first component of M. The matrix

$$
\begin{pmatrix}
\alpha^2 & \beta^2\\
2\alpha(\beta-\alpha) & 0\\
\alpha & -\alpha\\
\alpha^2(2\alpha-\beta) & (-2\alpha+\beta)\beta^2
\end{pmatrix}
$$

defined by the coefficients of these terms has rank 2, if $\alpha,\beta \neq 0$ and α,β. This implies the result for M.

It follows that

$$
\frac{\overline{\mathcal{E}}_{\overline{u}}}{RT(g,U) + <\lambda_1,u_4>\overline{\mathcal{E}}_{\overline{u}}}
$$

is generated by $[N,0,0,0]$,$[1,0,0,0]$,$[0,1,0,0]$ and $[0,0,1,0]$ as a vector space over $\mathbf{R}$, if $\alpha,\beta \neq 0$ and $\alpha \neq \beta$.

Step 2: *Let* $R := [< N^2, N\lambda_1, \lambda_1^2, \Delta, u_4 >, \mathcal{M}, \mathcal{M}, \mathcal{E}]$. *Then* $R/T(g,U)$ *is generated by* $[N\lambda_1,0,0,0]$, $[Nu_4,0,0,0]$,$[0,\lambda_1,0,0]$ *and* $[0,u_4,0,0]$ *as an* $\mathcal{E}_{\lambda_1 u_4}$*- module.*

By the preparation theorem it follows from the result of step 1 that

$$\frac{\bar{\mathcal{E}}_{\mathbf{U}}}{RT(g,U)}$$

is generated by $[N,0,0,0]$,$[1,0,0,0]$,$[0,1,0,0]$ and $[0,0,1,0]$ as an $\mathcal{E}_{\lambda_1 u_4}$-module. Now suppose $\phi \in R$. ϕ can be represented as

$$\phi = \phi_1[N,0,0,0] + \phi_2[1,0,0,0] + \phi_3[0,1,0,0] + \phi_4[0,0,1,0] + \mu\,,$$

where $\phi_i \in \mathcal{E}_{\lambda_1 u_4}$ $(i = 1,\ldots,4)$ and $\mu \in RT(g,U)$. Since $RT(g,U) \subset R$ — compare the list of generators for $RT(g,U)$ — this implies

$$\phi_1[N,0,0,0] + \phi_2[1,0,0,0] + \phi_3[0,1,0,0] + \phi_4[0,0,1,0] \in R\,.$$

This, in turn, implies $\phi_1,\phi_3,\phi_4 \in \mathcal{M}_{\lambda_1 u_4}$ and $\phi_2 \in \mathcal{E}_{\lambda_1 u_4}\{\lambda_1^2, u_4\}$. Therefore $R/RT(g,U)$ is generated as an $\mathcal{E}_{\lambda_1 u_4}$-module by

$$[N\lambda_1,0,0,0],[Nu_4,0,0,0],[\lambda_1^2,0,0,0],[u_4,0,0,0]\,,$$

$$[0,\lambda_1,0,0],[0,u_4,0,0],[0,0,\lambda_1,0],[0,0,u_4,0]\,.$$

Recalling that

$$T(g,U) = RT(g,U) + \mathcal{E}_{\lambda_1 u_4}\left\{[\lambda_1^2,0,0,0],[u_4,0,0,0],[0,0,\lambda_1,0],[0,0,u_4,0]\right\}$$

yields the result.

Step 3: $T(g,U) = R$.

It remains to show that $R \subset T(g,U)$. The elements of $RT(g,U)$ give rise to relations between the generators of the $\mathcal{E}_{\lambda_1 u_4}$-module $R/T(g,U)$. In order to express these relations more conveniently some redundant generators are added yielding the following list:

$$[N^2,0,0,0],[N\lambda_1,0,0,0],[N\Delta,0,0,0],[Nu_4,0,0,0],[\Delta,0,0,0]\,,$$

$$[0,N,0,0],[0,\Delta,0,0],[0,\lambda_1,0,0],[0,u_4,0,0],[0,N^2,0,0],$$
$$[0,0,N,0],[0,0,\Delta,0],[0,0,N^2,0],[0,0,N\Delta,0],[0,0,0,1],$$
$$[0,0,0,N],[0,0,0,N^2],[0,0,0,\Delta].$$

Now consider the relations defined by the following elements of $RT(g,U)$: $Ng_1, \Delta g_1, \ldots, g_{12}$ (the generators of $RT(g,U)$, 18 relations) and

$$Ng_2\,,\Delta g_2\,,Ng_3\,,Ng_7\,,Ng_{10}\,,Ng_{11}\,,\Delta g_{11}\,,-\frac{1}{2}N(g_{10}-g_3)\,,$$
$$-\frac{1}{2}(g_{10}-g_3)\,,-\frac{1}{2}(g_{11}-g_2)\,,-\frac{1}{2}N(g_{11}-g_2)\,,-\frac{1}{2}\Delta(g_{11}-g_2)\,,g_4-g_3\,,$$
$$g_5-\Delta g_1\,,N(Ng_1-g_4)\,,\frac{1}{2}(3\Delta g_1-\Delta g_9)\,,\frac{1}{2}N(3\Delta g_1-\Delta g_9)\,,$$
$$g_8+\Delta g_{11}\,,Ng_7+g_8\,,Ng_{10}-g_{11}\,,N(g_2-\lambda_1 g_9)\,,$$
$$\frac{1}{2}N(3Ng_1-Ng_9)\,,\frac{1}{2}\Delta(3Ng_1-Ng_9)\,,K\quad\text{and}\quad L$$

(another 25 relations), where

$$K := g_2-\beta g_3-3\alpha Ng_1+2\alpha Ng_9$$

$$L := \beta(g_3+g_4+Ng_1)-2\alpha N(g_9-g_1)\,.$$

The generators and relations define a matrix with entries in $\mathcal{E}_{\lambda_1 u_4}$ — the relations correspond to its rows and the generators to its columns. In order to show $R\subset T(g,U)$ we can ignore terms in $\mathcal{M}_{\lambda_1 u_4}R$ by Nakayama's lemma. This simplifies the matrix, which is displayed in figure 5.1 — α and β being replaced by a and b, respectively — and yields the following: If $\alpha,\beta\neq 0$ and $\alpha\neq\beta$, this matrix has rank 18. [2] This implies

$$\frac{R}{T(g,U)}=0$$

and hence $R\subset T(g,U)$. □

The following propositions 3.5.4, 3.5.5 and 3.5.6 are devoted to proving that the tangent space $T(g,U)$ of lemma 3.5.3 is intrinsic with respect to

[2]This was checked using symbolic computation.

```
[a, 1, 0, 0, 0, b, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0]
[0, 0, a, 0, 0, 0, b, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0]
[0, a, 0, 0, 0, 0, 0, b, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
[0, 0, 0, a, 0, 0, 0, 0, b, 0, 0, 0, 0, 0, 0, 0, 0, 0]
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, a, 0, 0, 0, b, 0, 0, 0]
[0, 0, 0, 0, b, a, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0]
[a, 1, 0, 0, - b, a - b, 0, 1, 0, 0, -1, 0, 0, 0, -1, 0, 0, 0]
[0, 0, a - b, 0, 0, 0, - b, 0, 0, a, 0, -1, 0, 0, 0, -1, 0, 0]
[0, 0, 0, -1, 0, 0, 0, 0, -1, 0, 0, - b, a, 0, 0, a - b, 0, 0]
[0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, b, 0, 0, 0, a, 0, 0]
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, a - b, 0, 0, a, - b]
[3 a, 1, 0, 0, 0, 3 b, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0]
[0, 0, 3 a, 0, 0, 0, 3 b, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0]
[0, 3 a, 0, 0, 0, 0, 0, 3 b, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
[0, 0, 0, 3 a, 0, 0, 0, 0, 3 b, 0, 0, 0, 0, 0, 0, 0, 0, 0]
[0, 0, 0, 0, - 2 a + b, a - 2 b, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0]
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, a - 2 b, 0, 0, 0, - 2 a + b, 0, 0, 0]
[0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 3 b, 0, 0, 0, a, 0, 0]
[0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, a, 0, 0, b, 0, 0]
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, a, 0, 0, 0, b]
[0, 0, b, 0, 0, 0, 0, 0, 0, a, 0, 0, 0, 0, 0, 1, 0, 0]
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, b, 0, 0, a, 0]
[0, 0, - 2 a + b, 0, 0, 0, 0, 0, 0, a - 2 b, 0, 0, 0, 0, 0, 1, 0, 0]
[0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, a - 2 b, 0, 0, - 2 a + b, 0, 0]
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, a - 2 b, 0, 0, 0, - 2 a + b]
[0, 0, a, 0, 0, 0, 0, 0, 0, b, 0, 0, 0, 0, 0, 0, 0, 0]
[0, 0, 0, 0, a, b, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, b, 0, 0, 0, a, 0, 0, 0]
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, b, 0, 0, a, 0, 0]
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, b, 0, 0, 0, a]
[a, 1, 0, 0, - 2 b, - b, 0, 0, 0, 0, -1, 0, 0, 0, -2, 0, 0, 0]

[0, 0, - a, 0, 0, 0, - 2 b, 0, 0, a, 0, -2, 0, 0, 0, -1, 0, 0]
[0, 0, [illegible], 0, 0, 0, 0, 0, 0, 2 b - a, 0, 0, 2, 0, 0, 1, 0, 0]
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0]
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0]
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2 a - 3 b, 0, 0, a, - 2 a]
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, a, 0, 0, 2 a, - b]
[0, 0, - 2 a + b, -1, 0, 0, 0, 0, 0, a - 2 b, 0, 0, a - 2 b, 0, 0, 1 + 2 a - b
, 0, 0]
[0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, a, 0, 0, b, 0, 0]
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0]
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0]
  2          2
[a , - a, 0, b, - b , 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]

[2 a (b - a), 3 b, 0, 0, 0, 0, 0, 2 b, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
```

Figure 5.1: The matrix defined by relations between the generators of $R/T(g,U)$

the group of $\mathbf{D}_4$- equivalences. This implies that the module $P(g)$ of higher-order terms contains $T(g,U)$, which is the content of lemma 3.5.7 below.

In order to show that $T(g,U)$ is intrinsic, it is necessary to consider the effect of $\mathbf{D}_4$-equivalences on a germ g by explicit coordinate changes.

Let $e = (S, X, \Lambda)$ be a $\mathbf{D}_4$-equivalence, where $X = [a,b,c,d]$, $a,b,c,d \in \mathcal{E}_{\overline{U}}$, $a(0) > 0$ and

$$\Lambda(\lambda_1,\lambda_2) = (\Lambda_1(\lambda_1,u_4), \lambda_2\Lambda_2(\lambda_1,u_4)),$$

where $\Lambda_1 \in \mathcal{M}_{\lambda_1 u_4}$, $\Lambda_2 \in \mathcal{E}_{\lambda_1 u_4}$.

The following list contains notation for some invariants in $\mathcal{E}_{\overline{U}}$, which will be used frequently below.

$$\begin{aligned}
\tilde{N}_1 &:= a^2N + b^2N\Delta + c^2Nu_4 + d^2N\Delta u_4 - 2ab\Delta - 2cd\Delta u_4\,, && (5.1)\\
\tilde{N}_2 &:= 2bcN + 2adN - 2ac - 2bd\Delta\,, && (5.2)\\
D_1 &:= a^2 + b^2\Delta + c^2u_4 + d^2\Delta u_4 - 2abN - 2cdNu_4\,, && (5.3)\\
D_2 &:= 2bc\Delta + 2ad\Delta - 2acN - 2bdN\Delta\,, && (5.4)\\
\Delta_1 &:= D_1^2\,, && (5.5)\\
\Delta_2 &:= D_1D_2\,, && (5.6)\\
\Delta_3 &:= D_2^2\,, && (5.7)\\
\tilde{\Delta}_1 &:= \Delta\Delta_1 + u_4\Delta_3\,, && (5.8)\\
\tilde{\Delta}_2 &:= 2\Delta_2\,. && (5.9)
\end{aligned}$$

Proposition 3.5.4 *Let X and Λ be defined as above and let $\tilde{N} := N \circ X$, $\tilde{\Delta} := \Delta \circ X$, $\tilde{\lambda}_1 := \lambda_1 \circ \Lambda$, $\tilde{u}_4 := u_4 \circ \Lambda$. Then the following formulae hold:*

$$\begin{aligned}
\tilde{N} &= \tilde{N}_1 + \delta\lambda_2\tilde{N}_2\,, && (5.10)\\
\tilde{\Delta} &= \tilde{\Delta}_1 + \delta\lambda_2\tilde{\Delta}_2\,, && (5.11)\\
\tilde{\lambda}_1 &= \Lambda_1(\lambda_1,u_4)\,, && (5.12)\\
\tilde{u}_4 &= u_4\Lambda_2^2(\lambda_1,u_4)\,. && (5.13)
\end{aligned}$$

Proof. The results are obtained by straightforward calculations. □

Proposition 3.5.6 *Let $p \in \mathcal{E}_{\overline{u}}$ and let X and Λ be defined as above. Let $\bar{p} := p \circ (X, \Lambda)$. Then there exist germs $p_1, p_2 \in \mathcal{E}_{\overline{u}}$ such that*

$$\bar{p} = p(\bar{N}_1, \bar{\Delta}_1, \bar{\lambda}_1, \bar{u}_4) + \delta\lambda_2 p_1 + \Delta u_4 p_2 .$$

The point of this statement is that the first term in the expression for $\bar{p}$ depends on $\bar{N}_1, \bar{\Delta}_1, \bar{\lambda}_1, \bar{u}_4$ only — and not on $\bar{N}_2$ and $\bar{\Delta}_2$. This will be relevant in the proof of lemma 3.5.7 below.

Proof. Define

$$F(\alpha_1, \alpha_2, x, y, z, w) := p(x + \alpha_1, y + \alpha_2, z, w) .$$

Then

$$\begin{aligned} F(\alpha_1, \alpha_2, x, y, z, w) &= F(0, 0, x, y, z, w) \\ &\quad + \sum_{i=1}^{2} \alpha_i \frac{\partial F}{\partial \alpha_i}(0, 0, x, y, z, w) \\ &\quad + \sum_{i+j=2} H_{ij}(\alpha_1, \alpha_2, x, y, z, w)\alpha_1^i \alpha_2^j \end{aligned}$$

for some germs H_{ij}. Putting

$$h_i := \frac{\partial F}{\partial \alpha_i}(0, 0, x, y, z, w) \qquad (i = 1, 2)$$

and applying the last equation to $\alpha_1 = \delta\lambda_2 \bar{N}_2$, $\alpha_2 = \delta\lambda_2 \bar{\Delta}_2$, $x = \bar{N}_1$, $y = \bar{\Delta}_1$, $z = \bar{\lambda}_1$ and $w = \bar{u}_4$ yields

$$\begin{aligned} & p(\bar{N}_1 + \delta\lambda_2 \bar{N}_2, \bar{\Delta}_1 + \delta\lambda_2 \bar{\Delta}_2, \bar{\lambda}_1, \bar{u}_4) \\ &= p(\bar{N}_1, \bar{\Delta}_1, \bar{\lambda}_1, \bar{u}_4) \\ &\quad + \delta\lambda_2 \left(\bar{N}_2 h_1(\bar{N}_1, \bar{\Delta}_1, \bar{\lambda}_1, \bar{u}_4) + \bar{\Delta}_2 h_2(\bar{N}_1, \bar{\Delta}_1, \bar{\lambda}_1, \bar{u}_4) \right) \\ &\quad + \sum_{i+j=2} H_{ij}(\delta\lambda_2 \bar{N}_2, \delta\lambda_2 \bar{\Delta}_2, \bar{N}_1, \bar{\Delta}_1, \bar{\lambda}_1, \bar{u}_4) . \end{aligned}$$

Since the germs $H_{ij}(\delta\lambda_2 \bar{N}_2, \delta\lambda_2 \bar{\Delta}_2, \bar{N}_1, \bar{\Delta}_1, \bar{\lambda}_1, \bar{u}_4)$ are Γ-invariant, there exist germs $K_{ij}, L_{ij} \in \mathcal{E}_{\overline{u}}$ such that

$$H_{ij}(\delta\lambda_2 \bar{N}_2, \delta\lambda_2 \bar{\Delta}_2, \bar{N}_1, \bar{\Delta}_1, \bar{\lambda}_1, \bar{u}_4) = K_{ij}(\overline{u}) + \delta\lambda_2 L_{ij}(\overline{u}) .$$

Defining

$$
\begin{aligned}
p_1 &:= \tilde{N}_2 h_1(\tilde{N}_1, \tilde{\Delta}_1, \tilde{\lambda}_1, \tilde{u}_4) + \tilde{\Delta}_2 h_2(\tilde{N}_1, \tilde{\Delta}_1, \tilde{\lambda}_1, \tilde{u}_4) \\
&\quad + \Delta u_4 \sum_{i+j=2} \tilde{N}_1^i \tilde{N}_2^j L_{ij}\,, \\
p_2 &:= \sum_{i+j=2} \tilde{N}_1^i \tilde{N}_2^j K_{ij}
\end{aligned}
$$

yields the result, since

$$\bar{p} = p(\tilde{N}_1 + \delta\lambda_2 \tilde{N}_2, \tilde{\Delta}_1 + \delta\lambda_2 \tilde{\Delta}_2, \tilde{\lambda}_1, \tilde{u}_4)\,.$$

□

Proposition 3.5.6 *Let $p, q, r, s \in \mathcal{E}_{\mathbf{U}}$. Then*

$$\delta\lambda_2[p, q, r, s] = [\Delta u_4, u_4 r, \Delta q, p]\,.$$

Proof. The following calculation yields the result:

$$
\begin{aligned}
&\delta\lambda_2 \left(p \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + q\delta \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} + r\lambda_2 \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} + s\delta\lambda_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) \\
&= p\delta\lambda_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + q\Delta\lambda_2 \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} + r u_4 \delta \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} + s\Delta u_4 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.
\end{aligned}
$$

□

Lemma 3.5.7 *Let $g = [\lambda_1 + \alpha N, \beta, 1, 0]$, where $\alpha, \beta \neq 0$ and $\alpha \neq \beta$. Then*

$$P(g) \supset \left[< N^2, N\lambda_1, \lambda_1^2, \Delta, u_4 >, \mathcal{M}, \mathcal{M}, \mathcal{E} \right]\,.$$

Proof. Let

$$R := \left[< N^2, N\lambda_1, \lambda_1^2, \Delta, u_4 >, \mathcal{M}, \mathcal{M}, \mathcal{E} \right]\,.$$

By lemma 3.5.3 $R = T(g, U)$. It will be shown that R is intrinsic with respect to the group of $\mathbf{D}_4$-equivalences. Then a theorem of Gaffney [Gaf86] implies the result.

Let $e = (S, X, \Lambda)$ be a $\mathbf{D}_4$-equivalence as defined above,

$$I := \langle N^2, N\lambda_1, \lambda_1^2, \Delta, u_4 \rangle$$

and let $p, q, r, s \in \mathcal{E}_a$ such that $p \in I$ and $q, r \in \mathcal{M}$. It will be shown first that $e.h_i \in R$ for $i = 1, \ldots, 4$, where

$$h_1 = [p, 0, 0, 0], h_2 = [0, q, 0, 0], h_3 = [0, 0, r, 0], h_4 = [0, 0, 0, s].$$

Consider the effect of a coordinate change (X, Λ) on h_1:

$$\begin{aligned} h_1 \circ (X, \Lambda) &= (p \circ (X, \Lambda))[a, b, c, d] \\ &= \bar{p}[a, b, c, d]. \end{aligned}$$

Hence by proposition 3 5.5

$$\begin{aligned} h_1 \circ (X, \Lambda) &= p(\tilde{N}_1, \tilde{\Delta}_1, \tilde{\lambda}_1, \tilde{u}_4)[a, b, c, d] \\ &\quad + \delta\lambda_2 p_1[a, b, c, d] \\ &\quad + \Delta u_4 p_2[a, b, c, d]. \end{aligned}$$

It follows from proposition 3.5.6 that the second and third term are in R. Consider the first term.

By formula 5.1

$$\tilde{N}_1 = a^2 N + m,$$

where $m \in I$. Hence $\tilde{N}_1^2 \in I$. Formula (5.8) implies $\tilde{\Delta}_1 \in I$. It is obvious that $\tilde{\lambda}_1^2 \in I$ and $\tilde{u}_4 \in I$. Also $\tilde{N}_1\tilde{\lambda}_1 \in I$, since $\tilde{\lambda}_1 \in \mathcal{M}_{\lambda_1 u_4}$. It follows that

$$p(\tilde{N}_1, \tilde{\Delta}_1, \tilde{\lambda}_1, \tilde{u}_4) \in I,$$

since $p \in I$. Hence $h_1 \circ (X, \Lambda) \in R$.

Now consider $h_2 \circ (X, \Lambda)$. Using formulae (5.3) and (5.4) a calculation shows that

$$h_2 \circ (X, \Lambda) = \bar{q}[b\Delta D_1 + cu_4 D_2, aD_1 + du_4 D_2, d\Delta D_1 + aD_2, cD_1 + bD_2],$$

where $\bar{q} = q \circ (X, \Lambda)$. By propositions 3.5.5 and 3.5.6 it suffices to consider

$$q(\tilde{N}_1, \tilde{\Delta}_1, \tilde{\lambda}_1, \tilde{u}_4)[b\Delta D_1 + cu_4 D_2, aD_1 + du_4 D_2, d\Delta D_1 + aD_2, cD_1 + bD_2].$$

Inspection of this expression shows that it is sufficient to prove that

$$q(\bar{N}_1, \bar{\Delta}_1, \bar{\lambda}_1, \bar{u}_4) a D_1 \in \mathcal{M} .$$

This, however, follows immediately, since $\bar{N}_1, \bar{\Delta}_1, \bar{\lambda}_1, \bar{u}_4 \in \mathcal{M}$ by formulae (5.1), (5.8),(5.12) and (5.13), and since $q \in \mathcal{M}$. Hence $h_2 \circ (X, \Lambda) \in R$.

Now consider $h_3 \circ (X, \Lambda) \in R$. A calculation yields

$$h_3 \circ (X, \Lambda) = \bar{r}[u_4 c \Lambda_2, u_4 d \Lambda_2, a \Lambda_2, b \Lambda_2] ,$$

where $\bar{r} = r \circ (X, \Lambda)$. Since $\bar{r} u_4 c \Lambda_2 \in I$ and $\bar{r} \in \mathcal{M}$, it follows that $h_3 \circ (X, \Lambda) \in R$.

For $h_4 \circ (X, \Lambda)$ one obtains

$$\begin{aligned} h_4 \circ (X, \Lambda) \quad = \quad & \bar{s}[D_1 \Lambda_2 \Delta u_4 d + D_2 \Lambda_2 u_4 a, D_1 \Lambda_2 u_4 c + D_2 \Lambda_2 u_4 b, \\ & D_1 \Lambda_2 \Delta b + D_2 \Lambda_2 u_4 c, D_1 \Lambda_2 a + D_2 \Lambda_2 u_4 d] , \end{aligned}$$

where $\bar{s} = s \circ (X, \Lambda)$, showing that $h_4 \circ (X, \Lambda) \in R$.

So far it has been shown that $h = [p, q, r, s] \in R$ implies $h \circ (X, \Lambda) \in R$. It remains to show that $h \in R$ implies $S_i h \in R$ for $i = 1, \ldots, 8$, where S_i are the generators of $\overleftrightarrow{\mathcal{E}}_{x\lambda}(\Gamma)$. (Compare proposition 3.3.5.) This can easily be verified by looking at the multiplication table 4.1 in subsection 3.4. It follows that R is intrinsic. □

Proof of theorem 3.5.1: Consider a bifurcation $h = [p, q, r, s]$ satisfying the recognition conditions $p = 0$, $q \neq 0$, $p_N \neq 0$, $p_N - q \neq 0$, $p_{\lambda_1} \neq 0$ and $r \neq 0$. Let $e = (S, X, \Lambda)$ be a $\mathbf{D}_4$-equivalence as above and let

$$a_0 := a(0) , \quad l_{10} := (\Lambda_1)_{\lambda_1}(0, 0) \quad \text{and} \quad m_0 := \Lambda_2(0, 0) .$$

By lemma 3.5.7

$$P(g) \supset \left[< N^2, N\lambda_1, \lambda_1^2, \Delta, u_4 >, \mathcal{M}, \mathcal{M}, \mathcal{E} \right] .$$

Working modulo $P(g)$ the germ h reduces to

$$\hat{h} := [p_{\lambda_1} \lambda_1 + p_N N, q, r, 0] .$$

Again working modulo $P(g)$ it is easy to check using formulae (5.1), (5.12) and table (4.1) that $e.h$ reduces to

$$[p_{\lambda_1} a_0 l_{10}\lambda_1 + p_N a_0^3 N, q a_0^3, r a_0 m_0, 0]\,. \tag{5.14}$$

Let $\epsilon_0 := \mathrm{sg} p_{\lambda_1}$ and $\epsilon_1 := \mathrm{sg} q$. The scaling equivalence defined by

$$a_0 := |q|^{-\frac{1}{3}}, \quad l_{10} := |p_{\lambda_1}|^{-1}|q|^{\frac{1}{3}} \quad m_0 := r^{-1}|q|^{\frac{1}{3}}$$

(5.14) becomes

$$\left[\epsilon_0\lambda_1 + \frac{p_N}{|q|}N, \epsilon_1, 1, 0\right].$$

Define $\alpha := p_N/|q|$. This shows that all bifurcations h satisfying the conditions in the theorem are $\mathbf{D}_4$-equivalent to the normal form g.

It follows by proposition 3.4.5 that

$$\begin{aligned} T_e(g) \;&=\; T(g,U) + \mathbf{R}\,\{[\lambda_1 + \alpha N, \epsilon_1, 1, 0], [\lambda_1 + 3\alpha N, 3\epsilon_1, 1, 0], \\ &\qquad [1,0,0,0], [\lambda_1, 0, 0, 0], [0,0,1,0]\}\,. \end{aligned}$$

A short calculation shows that

$$\begin{aligned} T_e(g) \;&=\; \left[< N^2, N\lambda_1, \lambda_1^2, \Delta, u_4 >, \mathcal{M}, \mathcal{M}, \mathcal{E}\right] \\ &\quad + \mathbf{R}\,\{\alpha[N,0,0,0] + \epsilon_1[0,1,0,0], [1,0,0,0], [\lambda_1,0,0,0], [0,0,1,0]\}\,. \end{aligned}$$

Hence $T_e(g)$ and therefore g is of $\mathbf{D}_4$-codimension 1.

To prove the last statement note first that for $h = [p,q,r,s]$ to be a bifurcation, it has to satisfy $p = 0$. If one of the degenaracy conditions is not satisfied, it follows by proposition 3.4.3 that

$$\mathrm{codim}^{\mathbf{D}_4}(h) \geq 2\,.$$

□

3.6 Geometrical description of the generic normal form

This subsection contains gyratory bifurcation diagrams for the generic normal form in theorem 3.5.1. These schematic diagrams contain the following information: The curves drawn in the diagrams represent the branches of the zero set of the normal form. The case $\epsilon_0 = \epsilon_1 = 0$ is considered — i. e. $g = [\lambda_1 + \alpha N, 1, 1, 0]$. Choosing other values for the signs ϵ_0 and ϵ_1 yields similar diagrams. The vertical coordinate corresponds to $N = x_1^2 + x_2^2$. The horizontal coordinate, which is denoted by s, parametrises a circle around the origin in parameter space given by $\lambda_1 = \cos s$ and $\lambda_2 = \sin s$. (This explains the term *gyratory*). There are three different cases to consider: $\alpha < 0, 0 < \alpha < 1$ and $\alpha > 1$.

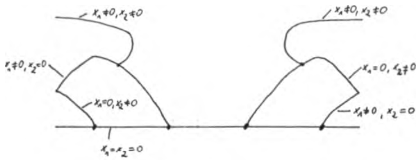

Figure 6.1: $\alpha < 0$

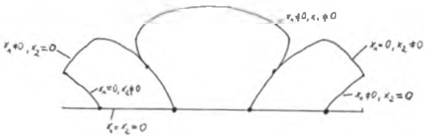

Figure 6.2: $0 < \alpha < 1$

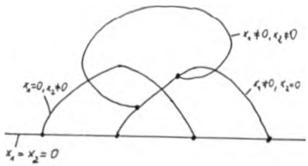

Figure 6.3: $\alpha > 1$

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